This is part 1 (of 3) of the weekly homework. It is due July 27 at the beginning of class.

SUMMARY.

- $\nabla f(x,y,z) = (f_x, f_y, f_z)$ **gradient**.
- $D_v f = \nabla f \cdot v$ **directional derivative**.
- **PDE** $F(f, f_x, f_t, f_{xx}, f_{tt}) = 0$.
- $f(x,y,z)$ function of three variables, $r(t)$ **curve**, $\frac{d}{dt} f(r(t)) = \nabla f(r(t)) \cdot r'(t)$ **chain rule**.
- $n \cdot (x,y,z) = d = n \cdot (x_0,y_0,z_0)$ **tangent plane** to $f(x,y,z) = c$ at point (x_0,y_0,z_0).
- $\nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = 0$ **tangent plane** to $f(\vec{x}) = c = f(\vec{x}_0)$ (vector notation).
- **Clairot**: $f_{xy} = f_{yx}$ for smooth functions.
- "Gradients are orthogonal to level curves rsp. level surfaces."
- $L(x,y) = f(x_0,y_0) + a(x - x_0) + b(y - y_0)$ **linear approximation** of $f(x,y)$ at (x_0,y_0).
- **Tangent line** $ax + by = d$ with $a = f_x(x_0,y_0), b = f_y(x_0,y_0)$, $d = ax_0 + by_0$.
- **Estimate** $f(x,y)$ by $L(x,y)$ near $f(x_0,y_0)$.
- Vector notation: $L(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0)$
- Punch line: $L(\vec{x})$ is close to $f(\vec{x})$ near $\vec{x}_0$ but a simple linear function. Many physical laws are actually linear approximations to more complicated laws.

Homework Problems

1) (4 points)
a) Sketch a contour map of the function $f(x,y) = x^2 + 9y^2$.
b) Find the **gradient vector** $\nabla f = (f_x, f_y)$ of f at the point $(1,1)$ and draw it.
c) Find the tangent line $ax + by = d$ to the curve at $(1,1)$ and draw it.
d) Estimate $f(1.001, 0.999)$ using linear approximations.

Solution:
a) The level curves $x^2 + 9y^2 = c$ are ellipses.
b) $\nabla f(x,y) = (2x, 18y)$. At the point $(1,1)$, it is the vector $(2,18)$.
c) The line has the form $2x + 18y = d$, where d is obtained by plugging in the coordinates of the point: $d = 20$. The equation is $x + 9y = 10$.
d) The linear approximation is $L(x,y) = f(1,1) + (2,18)(x-1, y-1) = 10 + 2 \cdot 0.001 - 8 \cdot 0.001 = 10 - 0.006 = 9.9984$.

2) (4 points) The **ideal gas law** for a gas with fixed mass m at temperature T, pressure P and volume V is $\boxed{PV = mRT}$, where R is the gas constant. Show that $\frac{\partial P}{\partial V} \frac{\partial V}{\partial T} \frac{\partial T}{\partial P} = -1$.

Note: the relation $PV = mRT$ leads to functions $P(V,T) = mRT/V$. Analogously there are functions $T(V,P), V(P,T)$. $\frac{\partial P}{\partial V}$ is a short hand notation for $\frac{\partial P(V,T)}{\partial V}$.

Solution:
$P(V,T) = mRT/V$: $\frac{\partial P}{\partial V} = -mRT/V^2$.
$V(T,P) = mRT/P$: $\frac{\partial V}{\partial T} = mR/P$.
$T(V,P) = PV/(mR)$: $\frac{\partial T}{\partial P} = V/(mR)$.
Therefore (using the equation itself at the end):

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = (-mRT/V^2)(mR/P)(V/mR) = -mRT/VP = -1$$

3) (4 points) **Cobb and Douglas** found in 1928 empirically a formula $P(L,K) = bL^\alpha K^\beta$ for the total production P of an economic system as a function of the amount of labor L and the capital investment K. By fitting data, they got $b = 1.01, \alpha = 0.75, \beta = 0.25$. Verify that the function $P(L,K)$ satisfies the PDE $LP_L + KP_K = P$.

Solution:
Leave α, β first as constants and fill then the constants in later:
$P_L = b\alpha L^{\alpha-1}K^\beta$.
$P_K = b\beta L^\alpha K^\beta$.

$$LP_L + KP_K = b\alpha L^\alpha K^\beta + b\beta L^\alpha K^\beta = (\alpha+\beta)P$$

Since $\alpha + \beta = 1$, the relation is true.

4) (4 points)
a) (2) Find the linear approximation $L(x,y)$ of the function $f(x,y) = \sqrt{10 - x^2 - 5y^2}$ at $(2,1)$ and use it to estimate $f(1.95, 1.04)$.
b) (2) Find the directional derivative $D_v f(2,1) = \nabla f(2,1) \cdot \vec{v}$ with $\vec{v} = (-5,1)$.

Solution:
a) $f(2,1) = 1$, $\nabla(x,y) = (-2x,-10y)/(2f(x,y) = (-x,-5y) = (-2,-5)$. $L(x,y) = f(2,1) + \nabla f(2,1)(x-2,y) = 1 + (-2,-5)(x-2,y) = 1 - x + (1-\pi)y$ so that $L(1.95,1.04) = f(2,1) + \nabla f(2,1)(-0.05,0.04) = 1 + (-2,-5)(-0.05,0.04) = 1 + 0.10 - 0.2 = 0.90$ which can be compared with the actual value $f(1.95,1.04) = 0.88888$. Note that the linear approximation could be computed without taking square roots.
b) $(-2,-5)(-5,1) = 5$.

5) (4 points)
Find $f(0.01, 0.999)$ for $f(x,y) = \cos(\pi xy)y + \sin(x + \pi y)$

Solution:
Approximate $f(x,y)$ by $L(x,y) = f(0,1) + \nabla f(0,1) \cdot (x-0, y-1)$ and then compute $L(0+0.01, 1-0.0001)$. We have $\nabla f(x,y) = (-\sin(\pi xy)\pi y^2 + \cos(x+\pi y), -\sin(\pi xy)\pi xy + \cos(\pi xy) + \pi\cos(x+\pi y)$ so that $\nabla f(0,1) = (-1, 1-\pi)$ Finally $L(x,y) = -1 + f_x(0,1)x + f_y(0,1)(y-1) = -1 - x - \pi y$ and therefore $L(0.01, 1-0.0001) = -1 - 0.01 + \pi 0.0001$.

Challenge Problems

(Solutions to these problems are **not** turned in with the homework.)

1) Verify that the **N-wave** function $f(t,x) = \frac{x}{t}\frac{\frac{1}{t}e^{-x^2/(4t)}}{1+\frac{1}{t}e^{-x^2/(4t)}}$ satisfies the PDE $f_t + ff_x = f_{xx}$ called **Burger's equation**.

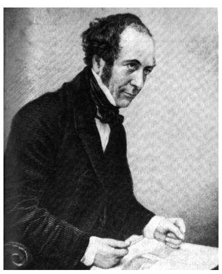
2) Verify that the **soliton** $f(t,x) = \frac{a^2}{2} \cosh^{-2}(\frac{a}{2}(x - a^2 t))$ is a solution of the **KDV equation** $f_t + 6ff_x + f_{xxx} = 0$.

3) The partial derivatives of the function $f(x,y) = (xy)^{1/3}$ exists at every point but the directional derivatives in all other directions don't exist at the point $(0,0)$. What is going on?

4) Extend the notion of "tangent plane" to 3-dimensional hyper-surfaces $f(x,y,z,w) = c$ in 4-dimensional space. For example, what is the tangent plane to the three-dimensional sphere $x^2 + y^2 + z^2 + w^2 = 1$ at the point $(x,y,z,w) = (1/2,1/2,1/2,1/2)$?

Remarks

(You don't need to read these remarks to do the problems.)



In the year 1834, while conducting experiments to determine the most efficient design for canal boats, a young Scottish engineer named **John Scott Russell** (1808-1882 see photo) made a remarkable scientific discovery at the Union Canal at Hermiston in Edinburgh. The effect described it in his "Report on Waves" has by the way been recreated in 1995 in the same channel by students of Heriot-Watt University (photo below).

> *"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation".*