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Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Total:

Welcome to the final exam. Please don't get started yet. We start all together at 9:00 AM after getting reminded about some formalities. You can fill out the attendance slip already. Also, you can already enter your name into the larger box above.

- You only need this booklet and something to write. Please stow away any other material and any electronic devices. Remember the honor code.
- Please write neatly and give details. Except for problems 2 and 3 we want to see details, even if the answer should be obvious to you.
- Try to answer the question on the same page. There is additional space on the back of each page. If you must, use additional scratch paper at the end. But put your final result near the question and box the final result.
- If you finish a problem somewhere else, please indicate on the problem page where we can find it.
- You have 180 minutes for this final exam.

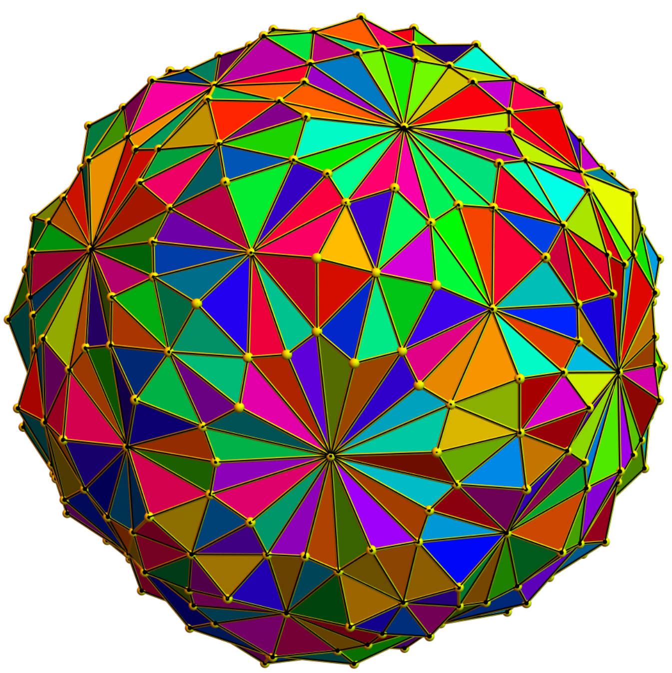


FIGURE 1. A two dimensional discrete sphere S.

Unit 39: Final Exam

Problems

Problem 39.1) (10 points):
In Figure 2 (see the next page for a larger version) you see a discrete two dimensional region G in which all triangles are oriented counter clockwise. The one-form F as a function on oriented edges is given in the picture. Answer the following questions and give reasons:

a) (2 points) The curl dF of F is a function on oriented triangles. What can you say about the sum over all the curl values dF in the graph G of Figure 2?

b) (2 points) Is F a gradient field $F = df$ for some function f on vertices?

c) (2 points) What is the sum of the natural divergence values d^*F on vertices?

d) (2 points) What was the name of the matrix $K = d^*d$ that acts on 0-forms. It has been defined more than 150 years ago.

e) (2 points) In Figure 1 on the front page, you saw a two-dimensional discrete sphere S. which plays the role of a closed surface $x^2+y^2+z^2=1$ in $\mathbb{R}^3$. Given a 1-form F, a function on oriented edges of S, what is the sum over all curls on S? The answer is a number but you have to justify the answer.

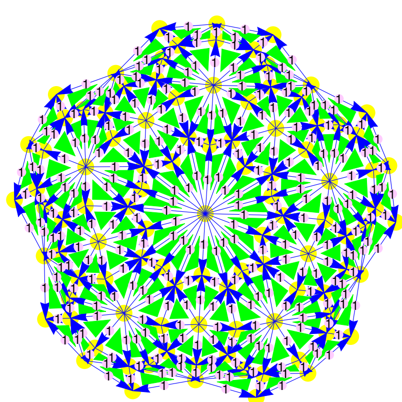
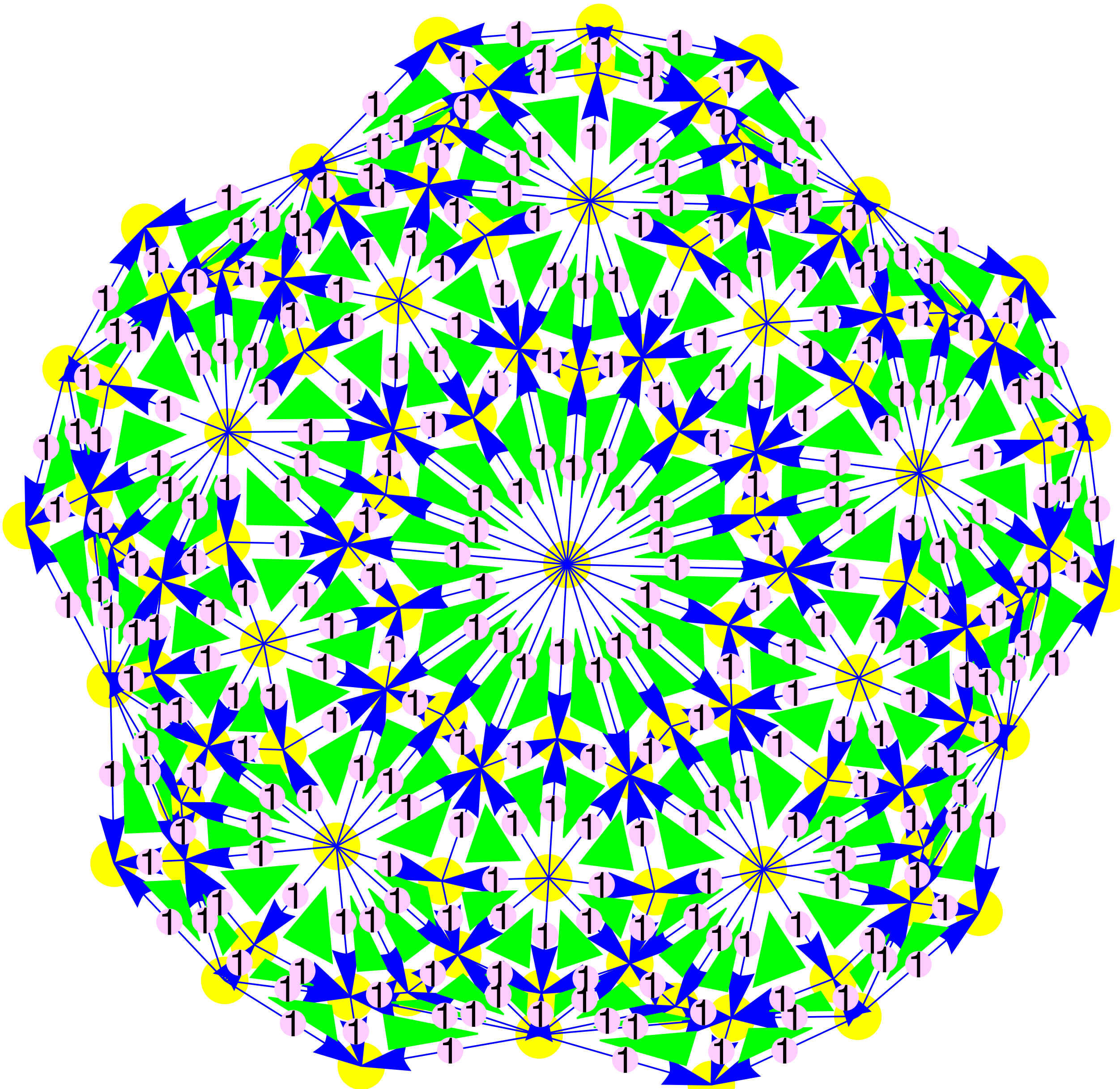


Figure 2.



An enlargement of Figure 2, showing the region G from problem 39.1.

Problem 39.2) (10 points) Each question is one point:

a) Albert Einstein used the notation $v_k w^k$ for two vectors v, w. It is today called "Einstein notation". What did Einstein mean, when he wrote $v_k w^k$?

b) If $S = r(R)$ is a two-dimensional surface parametrized by $r(u,v) = [x(u,v), y(u,v), z(u,v)]^T$, what is the relation between $|r_u \times r_v|$ and $\sqrt{\det(dr^T dr)}$?.

c) What is the Newton method used for? We have seen this numerical tool in a proof seminar.

d) What is the curvature of a circle with radius 20?

e) Define the 1×5 matrix $A = [1, 1, 1, 1, 1]$. One of the two matrices $A, B = A^T$ is row reduced. Which one?

f) What is the distortion factor of the coordinate change $\Phi(x, y) = (3x + y, x + y)$?

g) What is the numerical value of i^{22}, if $i = \sqrt{-1}$ is the imaginary unit?

h) What is the name of the differential equation $i\hbar \frac{d}{dt}\psi = K\psi$, where K is a matrix? It appears in a theory which also is called "matrix mechanics".

i) Why is the distance between two lines $r_1(t) = Q + tv$ and $r_2(t) = P + tw$ given by the formula $|(v \times w) \cdot PQ|/|v \times w|$?

j) You are given a Morse function f on a 2-torus and you count that f has 11 maxima and 11 minima. How many saddle points are there?

Herr Einstein wishes you good luck!

Problem 39.3) (10 points) Each question is one point:
In this problem, we work in hyperspace $\mathbb{R}^4$, where points have coordinates (x, y, z, w).

a) Write down the exterior derivative dF of the 2-form

$$F = x^2y^2z^2w^2dydz \ .$$

b) Write down the exterior derivative of the 3-form

$$F = x^2y^2z^2w^2dxdzdw \ .$$

c) Let G be the two-dimensional torus $x^2 + y^2 = 1, z^2 + w^2 = 1$ embedded in $\mathbb{R}^4$. What does the general Stokes theorem tell about $\iint_G F\ dS$, where F is the 2-form from a)?

d) What is $d^2F = ddF$, where F is the 2-form given in a)?

e) What is $d^2F = ddF$, where F is the 3-form given in b)?

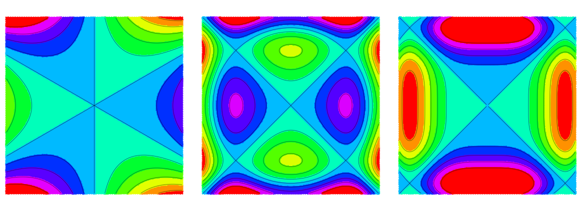
f) A $(1, 1)$ tensor on $\mathbb{R}^4$ can be interpreted as a 4×4 ....

g) A $(0, 1)$ tensor on $\mathbb{R}^4$ can also be interpreted as a ....

h) Is grad(grad(f)) defined if f is a function?

i) Does div(div(F)) make sense for any field F?

j) You see 3 contour maps of functions f, g and h of two variables. One of them is not Morse. Which one? The first the second or the third?



Problem 39.4) (10 points):
a) (3 points) Parametrize the line L which contains the points

$$A = (3,2,1), \quad B = (3,3,2) \ .$$

b) (3 points) Given the additional point $P = (3,3,3)$, find the distance between P and L.

c) (4 points) Write down the equation $ax + by + cz = d$ of the plane containing L and P.

Problem 39.5) (10 points):
a) (6 points) Find all the critical points of the function

$$f(x,y) = x^7 - 7x + xy - y$$

and classify them using the second derivative test.

b) (2 points) The island theorem told us that the number of maxima plus the number of minima minus the number of saddle points of f is 1 on an island. In the current case this fails. Why does this not contradict the island theorem?

c) (2 points) Does the function f have a global maximum or a global minimum?

Problem 39.6) (10 points):
a) (7 points) Use the Lagrange method to find the minimum of the function

$$f(x,y,z,w) = x^2 + 2y^2 + 3z^2 + w^2$$

under the constraint

$$g(x,y,z,w) = x + y + z + w = 17.$$

b) (3 points) You saw in a) that in this case, the Lagrange equations are a system of linear equations for a couple of unknown. This can be written in matrix form as $AX = b$, where the vector X encodes the unknown quantities and b is a constant vector. What is the size of the matrix A?

Problem 39.7) (10 points):
a) (5 points) Find the tangent plane at the point $P = (3, 1, 3, -1)$ of the **hyper cone**
$$S = \{f(x, y, z, w) = x^2 + y^2 - z^2 - w^2 = 0\}$$
in $\mathbb{R}^4$.

b) (5 points) Write down the linearization $L(x, y, z, w)$ of $f(x, y, z, w)$ at $(3, 1, 3, -1)$.

Problem 39.8) (10 points):
Estimate the value $f(0.1, -0.02)$ for $f(x, y) = e^{x+y}$ using quadratic approximation $Q(x, y)$ at $(x_0, y_0) = (0, 0)$.

Problem 39.9) (10 points):

a) (6 points) Find the curve $r(t)$ which satisfies $r(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and $r'(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $r''(t) = \begin{bmatrix} 1 - \sin(t) \\ -4\sin(2t) \\ -9\sin(3t) \end{bmatrix}$.

b) (4 points) What is the curvature of the curve at the point $r(0)$?

Problem 39.10) (10 points):
Find the area of the region enclosed by the curve
$$r(t) = \begin{bmatrix} 3\cos(t) \\ 2\sin(t) + \cos(7t) \end{bmatrix},$$
where $0 \le t \le 2\pi$.

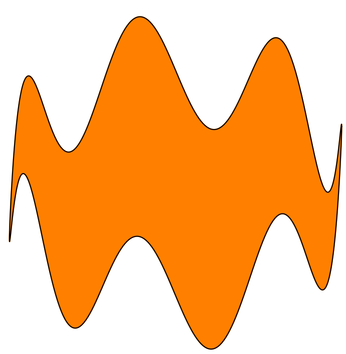


FIGURE 3. The region in problem 39.10.

Problem 39.11) (10 points):
Integrate

$$f(x,y,z) = \frac{e^{x^2+y^2+z^2}}{\sqrt{x^2+y^2+z^2}}$$

over the half avocado

$$E = \{4 \leq x^2+y^2+z^2 \leq 16, z \leq 0\} .$$

In other words, compute $\iiint_E f \, dV$.

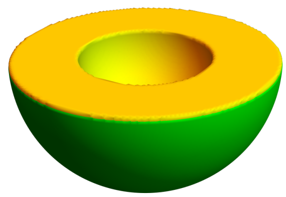


FIGURE 4. The avocado in problem 39.11.

Problem 39.12) (10 points):
Compute the line integral

$$\int_C F \cdot dr = \int_0^1 F(r(t)) \cdot r'(t) \, dt$$

of the vector field

$$F = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} 3x^2 + yz \\ 3y^2 + xz \\ 3z^2 + xy \end{bmatrix}$$

along the path C parametrized by

$$r(t) = \begin{bmatrix} \cos(7\pi t) e^{t(1-t)} \\ \sin(11\pi t) \\ e^{t(1-t)} \end{bmatrix}$$

from $t = 0$ to $t = 1$.

Problem 39.13) (10 points):
Find the line integral $\int_C F \cdot dr$ of the vector field

$$F(x,y) = \begin{bmatrix} y + x^4 \\ y^3 + y^4 \end{bmatrix}$$

along the boundary C of the hexagon region shown in the picture. The curve C is a closed polygon going counter clockwise from $(2,0)$ over $(1,2), (-1,2), (-2,0), (-1,-2), (1,-2)$ back to $(2,0)$.

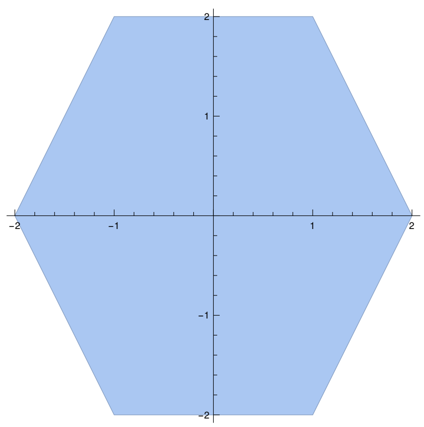


FIGURE 5. The hexagon in Problem 39.13.

Problem 39.14) (10 points):
Find the flux $\iint_S \text{curl}(F) \cdot dS$ of the curl of the vector field

$$F = \begin{bmatrix} x^7 \\ -x \\ \sin(z^2) + z^3 x \end{bmatrix}$$

through the surface S parametrized by

$$r(s,t) = \begin{bmatrix} (6 + 2\cos^2(s/2)\cos(t))\cos(2s) \\ 2\cos^2(s/2)\sin(t) + 2s \\ (6 + 2\cos^2(s/2)\cos(t))\sin(2s) \end{bmatrix}$$

with $0 \leq s \leq 7\pi/2$ and $0 \leq t < 2\pi$. **Hint:** The surface has two boundary curves obtained by looking at $s = 0$ or $s = 7\pi/2$. We don't tell you the orientation of the larger curve

$$r_1(t) = r(0,t) = [6 + 2\cos(t), 2\sin(t), 0]^T$$

is but you should know that the smaller curve

$$r_2(t) = r(7\pi/2, t) = [-6 - \cos(t), \sin(t) + 7\pi, 0]^T$$

is correctly oriented.

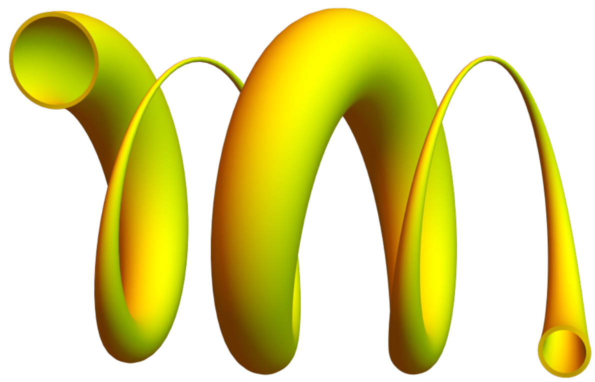


FIGURE 6. The surface S with two boundary circles in Problem 39.14.

Problem 39.15) (10 points):
Find the flux

$$\iint_S F \cdot dS$$

of the vector field

$$F = \begin{bmatrix} \sin(z) + y^3 + x \\ \sin(x) + z^3 + y \\ \sin(y) + x^3 + z \end{bmatrix}$$

through the boundary surface S of the solid E given in the picture. The solid is obtained by sculpuring a cube $-1 \leq x \leq 1, -1 \leq y \leq 1, -1 \leq z \leq 1$ of side length 2, by cutting away at each corner the points in distance less than 1 from that corner. In other words, we look at the points in the cube which have distance larger than 1 from any of the 8 corners. The surface S bounding the solid E is oriented outwards.

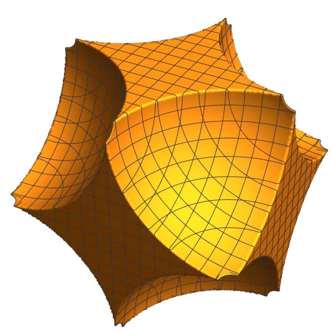


FIGURE 7. The solid given in Problem 39.15.