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Name:

LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Total:

Unit 38: Final Exam Practice

Problems

Problem 38P.1) (10 points):
a) Write down the Laplacian L of the complete graph with four vertices and find its eigenvalues and eigenvectors.
b) Solve the wave equation $f_{tt} = -Lf$ on the graph with initial position $f(0) = [2, 4, 3, 3]$.

Solution:
a) The Laplacian is

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

The eigenvalues are $0, 4, 4, 4$ as one can see that the matrix $A = L - 4$ has 3 eigenvalues 0 and an eigenvalue -4. The eigenvectors are $v_1 = [1, 1, 1, 1], v_2 = [-1, 1, 0, 0], v_3 = [-1, 0, 1, 0], v_4 = [-1, 0, 0, 1]$.
b) The initial condition is $3v_1 + v_2$. We can write down the closed form solution

$$f(t) = 3\cos(0t)[1, 1, 1, 1] + \cos(2t)[-1, 1, 0, 0]$$

Problem 38P.2) (10 points):

a) The set of all 800×600 gray scale pictures are a subspace of a large vector space $M(m,n)$ What is the dimension of this space?

b) The map $(x,y,z,w) \to (y,z,w,x)$ defines a linear transformation on $\mathbb{R}^4$. What is its determinant?

c) The set of all words in the alphabet $A - Z$ feature an addition $+$ given by concatenation. The zero element 0 is the empty word. What is the name of this algebraic structure?

d) Which of the three following people is not associated to an axiom system *Euclid*, *Peano*, *Jordan*.

e) What does the rank-nullity theorem tell for a $n \times m$ matrix?

f) If S is the eigenbasis of a $n \times n$ matrix A, what can you say about $S^{-1}AS$?

g) If A is a $n \times 1$ matrix and $A = QR$ is the QR-decomposition of A, what shape does the matrix R have?

h) What is the name of a matrix A with complex entries for which $\overline{A}^T A = 1$.

i) What is the determinant of a matrix $A \in SU(2)$?

j) Which of the fundamental forces are associated to $SU(2)$. The electromagnetic, the weak, the strong or the gravitational force?

Solution:

a) $800 \cdot 600 = 480'000$.

b) It is a permutation matrix with three upcrossings. The determinant is -1.

c) It is a monoid.

d) Jordan.

e) The rank plus the nullity is is equal to m.

f) It is a diagonal matrix.

g) It is a 1×1 matrix.

h) It is called unitary.

i) 1, by definition.

j) The weak force.

Problem 38P.3) (10 points):
We look for 4 numbers x, y, z, w. We know their sum is 20 and that their "super sum" $x - y + z - w$ is 10. As a matter of fact these two equations form a system $Ax = b$ which defines a 2-dimensional plane V in 4-dimensional space.
a) (6 points) Find the solution space of all these numbers by row reducing its augmented matrix $B = [A|b]$ carefully.

$$B = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 20 \\ 1 & -1 & 1 & -1 & 10 \end{array}\right] .$$

b) (4 points) Find two linearly independent vectors which are perpendicular to the kernel of A.

Solution:
a) Row reduction leads to

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 15 \\ 0 & 1 & 0 & 1 & 5 \end{array}\right]$$

The solution is

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} .$$

b) To be perpendicular to the kernel of A means being in the image of A^T. So, the rows of A are the solution

$$\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \} .$$

Problem 38P.4) (10 points):
People on social media have been in war about expressions like $\boxed{2x/3y - 1 \text{ if } x = 9 \text{ and } y = 2}$. Computers and humans disagree: most humans get 2, while most machines return 11. A psychologist investigates whether the size of the numbers influences the answer and asks people. This needs data fitting: using the least square method, find those a and b such that

$$\frac{ax}{3y} - b = 2$$

best fits the data points in the following table:

x	y
9	3
6	1
-3	1
0	1

Solution:
We write down the systems of equations (don't leave out any of the equations)
$a - b = 2$
$2a - b = 2$
$-a - b = 2$
$-b = 2.$
The matrix is

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \\ -1 & -1 \\ 0 & -1 \end{bmatrix}$$

We have $b = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$. The solution $\vec{x} = (A^T A)^{-1} A^T b$ is $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$. This is not a surprise as $a = 0, b = -2$ exactly solves the problem.

Problem 38P.5) (10 points):
Let $\vec{x} = \begin{bmatrix} v \\ e \\ f \end{bmatrix}$ denote the number of vertices, edges and faces of a polyhedron. During a **Barycentric refinement**, this vector transforms as

$$A\vec{x} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & 6 \end{bmatrix} \vec{x} \, .$$

a) (5 points) Verify that $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ are eigenvectors of A and find their eigenvalues.
b) (5 points) Write down a closed form solution of the discrete dynamical system $\vec{x}(t+1) = A\vec{x}(t)$ with the initial condition $\begin{bmatrix} v \\ e \\ f \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \\ 6 \end{bmatrix}$.

Solution:
a) Just compute $A\vec{v}_1$ and see that it is a multiple of $\vec{v}_1$. Similarly, do that with $A\vec{v}_2$ and $A\vec{v}_3$. The eigenvalues are $1, 2, 6$. b) Write

$$\begin{bmatrix} 7 \\ 12 \\ 6 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

giving $c_1 = 1$, $c_2 = 3$ and $c_3 = 3$. We can now write down the closed form solution

$$1^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2^t \cdot 3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 6^t \cdot 3 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} .$$

Problem 38P.6) (10 points):
The **Arnold cat map** is $T\vec{v} = A\vec{v}$ where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} .$$

It is an icon of chaos theory.
a) (2 points) What is the characteristic polynomial of A?
b) (2 points) Find the eigenvalues of A.
c) (2 points) Find the eigenvectors of A.
d) (2 points) Is the discrete dynamical system defined by A asymptotically stable or not?
e) (2 points) Write down an orthogonal matrix S and a diagonal matrix B such that $B = S^{-1}AS$.

Solution:
a) $\lambda^2 - 3\lambda + 1 = 0$.
b) $\lambda_1 = \frac{3+\sqrt{5}}{2}$, $\lambda_2 = \frac{3-\sqrt{5}}{2}$. c) The eigenvectors are $\begin{bmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{bmatrix}$ and $\begin{bmatrix} -1 \\ \frac{\sqrt{5}+1}{2} \end{bmatrix}$.
d) No, the absolute value of the eigenvalues is bigger than 1.
e) The matrix S is the matrix in which normalized eigenvectors are put as columns. The matrix B is the diagonal matrix with eigenvalues in the diagonal.

Problem 38P.7) (10 points):
The following configuration is called the "Beacon Oscillator" in the **Game of Life**.

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) (2 points) What is the rank and the nullity of A?
b) (4 points) Find a basis for the kernel and a basis for the image of A.
c) (4 points) The following matrix is called the "glider configuration" in the **Game of life**.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} .$$

Find the inverse of A using row reduction.

Solution:
a) The rank is 4, the nullity is 2.
b) A basis of the kernel is $\mathcal{B} = \{e_1, e_6\}$. A basis for the image are the 4 middle columns of the matrix.
c) $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

Problem 38P.8) (10 points):

Remember to give computation details. Answers alone can not be given credit.

a) (2 points) The following matrix displays the solution of the Cellular automaton 10. Find its determinant

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} .$$

b) (2 points) Find the determinant of

$$B = \begin{bmatrix} 0 & 0 & 3 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} .$$

c) (2 points) Find the determinant of

$$C = \begin{bmatrix} 1 & 2 & 3 & 8 & 8 \\ 4 & 5 & 0 & 8 & 8 \\ 6 & 0 & 0 & 8 & 8 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} .$$

d) (2 points) Find the determinant of

$$D = \begin{bmatrix} 11 & 2 & 3 & 2 & 1 \\ 1 & 12 & 3 & 2 & 1 \\ 1 & 2 & 13 & 2 & 1 \\ 1 & 2 & 3 & 12 & 1 \\ 1 & 2 & 3 & 2 & 11 \end{bmatrix} .$$

e) (2 points) Find the determinant of $E = 2Q + 5Q^{-1} + 7I$: (you can leave it in terms of eigenvalues of the basic circulant matrix Q you have seen. No simplifications are required):

$$E = \begin{bmatrix} 7 & 2 & 0 & 0 & 5 \\ 5 & 7 & 2 & 0 & 0 \\ 0 & 5 & 7 & 2 & 0 \\ 0 & 0 & 5 & 7 & 2 \\ 2 & 0 & 0 & 5 & 7 \end{bmatrix} .$$

Solution:
a) Patterns, 6 upcrossings, det=1
b) Patterns or row reduction $det = -120$.
c) Partitioned. $det = -450$.
d) Build $B = A - 10$ which has eigenvalues $0, 0, 0, 0, 9$, so that B has eigenvalues $10, 10, 10, 10, 19$ which is 190000
e) The determinant is the product of the eigenvalues $\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5$ where $\lambda_k = 7 + 2\exp(2\pi ik/5) + 5\exp(-2\pi ik/5)$.

Problem 38P.9) (10 points):
Find the general solution to the following differential equations:
a) (1 point)

$$f'(t) = 1/(t+1)$$

b) (1 point)

$$f''(t) = e^t + t$$

c) (2 points)

$$f''(t) + f(t) = t + 2$$

d) (2 points)

$$f''(t) - 2f'(t) + f(t) = e^t$$

e) (2 points)

$$f''(t) - f(t) = e^t + \sin(t)$$

f) (2 points)

$$f''(t) - f(t) = e^{-3t}$$

Solution:
a) $\log(t+1) + C$
b) $e^t + t^3/6 + C_1 t + C_2$.
c) $C_1\cos(t) + C_2\sin(t) + t + 2$
d) $C_1e^t + C_2te^t + t^2e^t/2$
e) $C_1e^t + C_2e^{-t} + te^t/2 - 1/2\sin(t)$
f) $C_1e^t + C_2e^{-t} + e^{-3t}/8$

Problem 38P.10) (10 points):

We consider the nonlinear system of differential equations

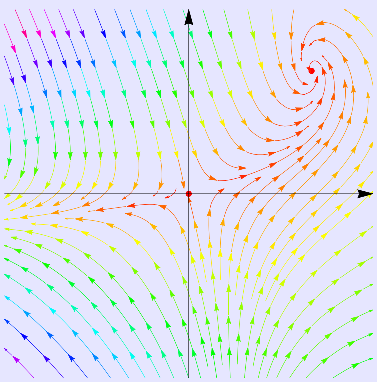
$$\begin{aligned} \frac{d}{dt}x &= x + y - xy \\ \frac{d}{dt}y &= x - 3y + xy \; . \end{aligned}$$

a) (2 points) Find the equilibrium points.

b) (3 points) Find the Jacobian matrix at each equilibrium point.

c) (3 points) Use the Jacobian matrix at an equilibrium to determine for each equilibrium point whether it is stable or not.

d) (2 points) Which of the diagrams A-D is the phase portrait of the system above?



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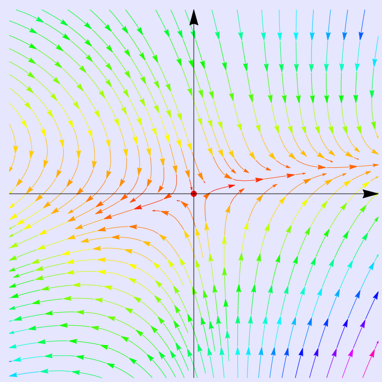

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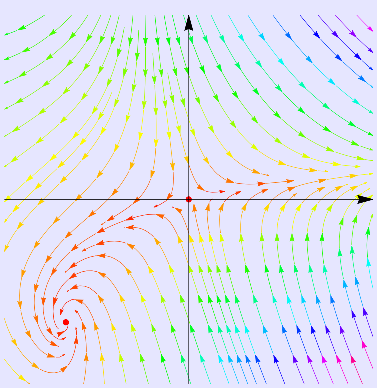

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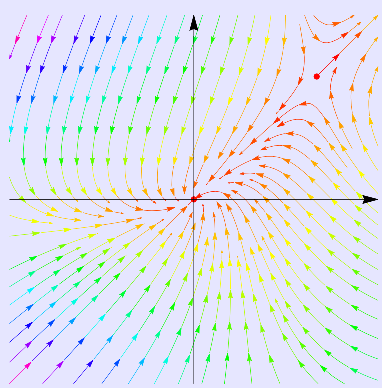

D

Solution:
a) The equilibrium points are $(0,0)$ and $(2,2)$.
b) The Jacobian matrices are

$$\begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 3 & -1 \end{bmatrix}$$

c) The first equilibrium point is unstable, the second is stable. d) We have phase portrait A.

Problem 38P.11) (10 points):
a) (6 points) Find the **Fourier series** of the function which is 1 if $|x| > 1$ and -1 else. We call it the **Pacific rim** function.

$$f(x) = \begin{cases} 1 & |x| > 1 \\ -1 & |x| \leq 1 \end{cases} .$$

b) (4 points) Find the value of the sum of the squares of all the Fourier coefficients of f.

Solution:
a) The function is even. It has a cos-series. We have

$$a_n = \frac{2}{\pi}\int_0^1 -\cos(nx)\,dx + \frac{2}{\pi}\int_1^\pi \cos(nx)\,dx = -4\sin(n)/(\pi n) .$$

and

$$a_0 = \frac{2}{\pi}\int_0^1 -1/\sqrt{2}\,dx + \frac{2}{\pi}\int_1^\pi 1/\sqrt{2}\,dx$$

which is $-\sqrt{2}/\pi + (\pi-1)\sqrt{2}/\pi$.
It could be simplified to $(\pi-2)\sqrt{2}/\pi$.
b) By Parseval, we know the sum of the squares $\sum_{n=0}^{\infty} a_n^2$ equal to $(2/\pi)\int_0^\pi 1^2\,dx = \boxed{2}$.

Problem 38P.12) (10 points):
a) Solve the system $f_t = 3f_{xx} - f + t$ with $f(0,x) = x$ on $[-\pi, \pi]$
b) Solve the system $f_t = 3f_{xx} - 9f_{xxxx}$ with $f(0,x) = x$.

Solution:
a) A particular solution which does not depend on x is $f(t) = t^2/2$. Now look at the homogeneous part. Since the operator $3D^2 - 1$ has eigenvalues $\lambda_n = -3n^2 - 1$, we have to look at the solution of $f_t = \lambda f$ which has solution $e^{-3n^2 t}$. The initial position is

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx) \; .$$

The solution of the heat equation is now

$$f_{hom}(t,x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} e^{(-3n^2-1)t} \sin(nx) \; .$$

The final solution is $f_{part}(x,t) + f_{hom}(t,x)$.
b) Since the operator $3D^2 - 9D^4$ has eigenvalues $\lambda_n = -3n^2 - 9n^4$ the solution is

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} e^{(-3n^2-9n^4)t} \; \sin(nx) \; .$$

Problem 38P.13) (10 points):
a) Solve the system $f_{tt} = 9 f_{xx}$ with $f(0,x) = x$ and $f_t(0,x) = 11\sin(88x)$.
b) Solve $f_{tt} = 9 f_{xx} - f_{xxxx}$ with $f_t(0,x) = x$.

Solution:
a) Since the operator $9D^2$ has eigenvalues $\lambda_n = -9n^2$, we have to look at the solution of the harmonic oscillator $f_{tt} = -9n^2 f$. The initial position is

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx) \ .$$

The solution of the wave equation for position is now

$$f_{pos} = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \cos(3nt) \ \sin(nx) \ .$$

The initial velocity is already a Fourier series (it has only one summand. Its solution is

$$f_{vel} = 11 \frac{\sin(3 \cdot 88t)}{3 \cdot 88} \sin(88x) \ .$$

The final solution is $f_{pos} + f_{vel}$.
b) The initial velocity is given as

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx) \ .$$

The function $\sin(nx)$ is still an eigenfunction of $9D^2 - D^4$ but the eigenvalue is $\lambda_n = -9n^2 - n^4$. The solution of the wave equation is now

$$f = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \frac{\sin(\sqrt{9n^2 + n^4} t)}{\sqrt{9n^2 + n^4}} \sin(nx) \ .$$