LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Unit 38: Checklist for Final

Glossary mostly since unit 28

- **linear space** X If f, g are in X, then $f + g, \lambda g$ are in X. Especially, 0 is in X.
- **linear map** $T(f + g) = T(f) + T(g), T(\lambda f) = \lambda T(f)$ and $T(0) = 0$.
- **diagonalization** possible if A is symmetric or normal or if spectrum is simple.
- **trace** $\text{tr}(A) =$ sum of diagonal entries $= \lambda_1 + \lambda_2 + \cdots + \lambda_n$.
- **determinant** $\det(A) =$ product of diagonal entries $= \lambda_1 \cdot \lambda_2 \cdots \lambda_n$.
- **diagonalization** $B = S^{-1}AS$, with diagonal B, where S contains eigenbasis of A.
- **Jordan normal form** $\lambda 1 + B$, where B has only super diagonal entries 1.
- **differential operator** like $p(D) = D^2 + 3D$ then $p(D)f = g$ reads $f'' + 3f' = g$.
- **homogeneous ODE** $p(D)f = 0$. Example: $f'' + 3f' = 0$.
- **inhomogeneous** ODE $p(D)f = g$. Example: $f'' + 3f' = \sin(t)$.
- **first order linear** $f' = \lambda f$, $f(t) = e^{\lambda t} f(0)$.
- **operator method** $((D - \lambda)^{-1} g)(t) = Ce^{\lambda t} + e^{\lambda x} \int_0^t e^{-\lambda s} g(s)\, ds$.
- **inner product** $\langle f, g\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)\, dx$, **length** $\sqrt{\langle f, f\rangle} = ||f||$.
- **Fourier series** $f(x) = a_0/\sqrt{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$.
- **Fourier basis** $1/\sqrt{2}, \cos(nx), \sin(nx)$ for piecewise smooth functions on $[-\pi, \pi]$.
- **Fourier coefficients** $a_0 = \langle f, 1/\sqrt{2}\rangle$, $a_n = \langle f, \cos(nx)\rangle$, $b_n = \langle f, \sin(nx)\rangle$.
- **odd case** use $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx$.
- **even case** use $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, dx$
- **Parseval identity** $a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = ||f||^2$
- **stability** $|\lambda_i| < 1$ for discrete systems and $\text{Re}(\lambda_i) < 0$ for continuous systems.
- **nonlinear differential equation** $x' = f(x, y), y' = g(x, y)$.
- **equilibrium points** points, where $f(x, y) = g(x, y) = 0$.
- **X-nullclines** $f(x, y) = 0$. **Y-nullclines** are curves, where $g(x, y) = 0$.
- **Jacobian** $\begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix}$.
- **heat equation** $f_t = \mu D^2 f$ solution example $\sum_{n=1}^{\infty} b_n e^{-n^2 \mu t} \sin(nx)$
- **wave** $f_{tt} = c^2 D^2 f$ solution $\sum_{n=1}^{\infty} b_n \cos(nct) \sin(nx)$ or $\sum_n \frac{b_n}{nc} \sin(nct) \sin(nx)$
- **generalized heat** $f_t = p(D)$ has solution $f(x, t) = \sum_{n=1}^{\infty} b_n e^{\lambda_n t} \sin(nx)$.
- **generalized wave** $f_{tt} = p(D)$ has solution $f(x, t) = \sum_{n=1}^{\infty} b_n \cos(\sqrt{-\lambda_n} t) \sin(nx)$ (position) or $\sum_n \frac{b_n}{\sqrt{-\lambda_n}} \sin(\sqrt{-\lambda_n} t) \sin(nx)$, with λ_n eigenvalue of $p(D)$ (velocity). Add both, if both position and velocity are given.
- **Parseval identity** The Pythagoras identity for Fourier $||f||^2 = a_0^2 + \sum_n a_n^2 + b_n^2$.

Key Points

- [ ] **Transformations** Relate a geometric transformation with matrix: look at columns!.
- [ ] **Determinants** Laplace, Row reduce, patterns, partitioned, triangular, eigenvalues.
- [ ] **Determinants** are compatible with matrix multiplication $\det(AB) = \det(A)\det(B)$.
- [ ] **Data fitting** First write down system, then use least square solution formula.
- [ ] **Diagonalization** is possible for normal matrices OR if all eigenvalues are different.
- [ ] **Similarity** see whether two matrices are similar.
- [ ] **Jordan normal form** is defined for all matrices.
- [ ] **QR decomposition** is defined for all matrices A with independent columns.
- [ ] **stability** Eigenvalues determine the asymptotic stability. $|\lambda| < 1$ or $\text{Re}(\lambda) < 0$.
- [ ] **closed form solution** use eigenbasis both in the discrete and continuous case.
- [ ] **basic differential equations** $\boxed{f' = \lambda f}$, $\boxed{f'' + c^2 f = 0}$.
- [ ] **Fourier basis** diagonalizes D^2. Watch even and odd case. For PDE's, use sin-series.
- [ ] **inhomogeneous cases** are solved by cookbook.
- [ ] **nonlinear systems** can be understood by analyzing equilibrium points.
- [ ] **Markov matrices** have a Perron-Frobenius eigenvector 1. Can be multiplied.

Skills

- [ ] Solve systems of linear equations. Find kernel and image.
- [ ] Understand linear spaces, linear maps. Distinguish whether linear or not.
- [ ] Find eigenvalues/eigenvectors. Use tricks of large kernels, sums of rows are constant.
- [ ] When are matrices similar? Criteria for similarity and contradicting it.
- [ ] Fit data with functions. Use the least square solution $(A^T A)^{-1} A^T b$ of $Ax = b$.
- [ ] Make QR decomposition of a matrix.
- [ ] Algebra of complex numbers: Add, subtract, multiply, take roots.
- [ ] Solve discrete dynamical systems $x(n+1) = Ax(n)$. Closed form.
- [ ] Solve continuous dynamical systems $x' = Ax$. Closed form.
- [ ] Solve differential equations $p(D)f = g$ by factoring p or using "cookbook".
- [ ] Decide stability for continuous and discrete dynamical systems.
- [ ] Analyze nonlinear systems: equilibrium points, null-clines, stability.
- [ ] Match phase space with system. Both linear and nonlinear.
- [ ] make Fourier synthesis of function $f(x)$ on $[-\pi, \pi]$.
- [ ] Know that Fourier basis diagonalizes D^2 or $p(D^2)$ like $D^2 - D^4 + 1$.
- [ ] Apply Parseval to relate Fourier coefficients with length $||f||$ of f.
- [ ] Solve heat type equations $f_t = p(D)f + g(t)$ with closed form solution.
- [ ] Solve wave type equations $f_{tt} = p(D)f + g(t)$ with closed form solution.
- [ ] Find projection onto the image of a matrix.

Closed form solutions

Partial differential equations like the **heat equation** $f_t = D^2 f$ or modifications like $f_t = (D^2 - D^4)f$ or $if_t = D^2 f$ or the **wave equation** $f_{tt} = D^2 f$ or modifications like $f_{tt} = (D^2 - D^4)f$ are solved with the Fourier basis $\sin(nx), \cos(nx), 1/\sqrt{2}$ which is an eigenbasis for D^2 or modifications like $p(D) = D^4 + 2D^6 + 4$. To do so, we use $\boxed{f'(t) = \lambda f(t)}$ with solution $f(t) = f(0)e^{\lambda t}$ or the **harmonic oscillator** $\boxed{f''(t) = -c^2 f(t)}$ with solution $f(0)\cos(ct) + f'(0)\sin(ct)/c$, where $c = \sqrt{-\lambda}$.

For discrete dynamical systems, write the initial condition as a linear combination of eigenvectors, then write down the solution $\sum_{k=1}^n c_k \lambda_k^t v_k$.

For continuous dynamical systems, write the initial condition as a linear combination of eigenvectors, then write down the solution $\sum_{k=1}^n c_k e^{\lambda t} v_k$.

Words of wisdom

- "Columns are the basis of an image"
- "Round and round you go with circular matrices."
- "Laplace row reduce ..."
- "If you feel tears, think shears".
- "PDE's are solved easily with Fourier!"
- "Odd functions provoke sins."
- "Even functions have a cause."
- "$f'(t) = \lambda f(t)$ is the mother of ODE's."
- "$f''(t) = -c^2 f(t)$ is the father of ODE's."
- "Oh, PDE, oh PDE, solved easily with Fourier."

Type of matrices

- projection dilations
- reflection dilations
- rotation dilations
- dilations
- shear dilations
- SU(2) matrices
- The magic matrix
- Circular matrices
- normal matrices $A^T A = AA^T$.

People

Since second midterm:

- **Fourier** (series)
- **Dirichlet (kernel)**
- **Parseval** (identity)
- **Pythagoras (tree)**
- **Schroedinger (cat)**
- **Arnold** (cat)
- **Grumpy** (cat)
- **Perron and Frobenius (maximal eigenvalue)**

- **Markov (matrix)**
- **Lorenz (attractor)**
- **Lyapunov (exponent)**
- **Goldbach (conjecture)**
- **Feigenbaum (universality)**

From Second midterm:

- **Murray** (system)
- **Euler** (Euler identity)
- **Grothendieck** (rising sea)
- **von Neumann** (wiggle)
- **Wigner** (wiggle)
- **Jordan** (Jordan normal form)
- **Laplace** (Laplace expansion)
- **Kac** (Can one hear a drum?)
- **Gordon-Webb-Wolpert** (No one can't!)
- **Leonardo Pisano** (Fibonacci Rabbits)
- **Lyapunov** (Lyapunov exponent)
- **Arnold** (cat map)
- **Cayley and Hamilton** (theorem)
- **Lorentz** (Lorentz system)

From First midterm:

- **Feynman** (examples)
- **Peano** (Peano axioms)
- **Euclid** (Axioms of geometry)
- **Gram** (QR)
- **Schmidt** (QR)
- **Gauss** (row reduction)
- **Jordan** (row reduction)
- **Hamilton** (quaternions)
- **Lagrange** (four square theorem)
- **Glashow, Salam, Weinberg** (electoweak unification)
- **Landau** (complexity)

Proof seminar

- **Axiom systems** What is a monoid. What is a group. What is a linear space.
- **Unitary matrices** What is $SO(3)$, $SU(2)$?
- **Complexity** What does $O(x^3)$ mean?
- **Raising sea** Why is it good to have a theory?
- **Spectra** Isospectral drums or graphs.
- **Golden mean** What is it? Where does it appear?
- **Chaos** What is the Lyapunov exponent?
- **Circular matrices** For Discrete Fourier transform.

- [ ] **Cookbook** Why does it work?
- [ ] **Dirichlet's proof** Know some ingredients like what done in HW
- [ ] **Applications of Fourier:** Sound, tomography, fast multiplication, number theory.

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