LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Unit 29: Fourier series

Lecture

29.1. It is convenient for applications to extend the linear space $C^\infty(\mathbb{T})$ of all smooth 2π periodic functions and consider the larger linear space $\mathcal{X}$ of **piecewise smooth periodic functions**. We can draw them functions on the interval $[-\pi, \pi]$. It contains functions as drawn in figure (1). We always draw functions in $\mathcal{X}$ as functions on $[-\pi, \pi]$ and do not insist that the left and right value agree as this just produces another jump when seeing as a function on the circle $\mathbb{T}$. In particular, we just write $f(x) = x$ for example, and draw it as a function on $[-\pi, \pi]$ then think of it 2π-periodically continued.

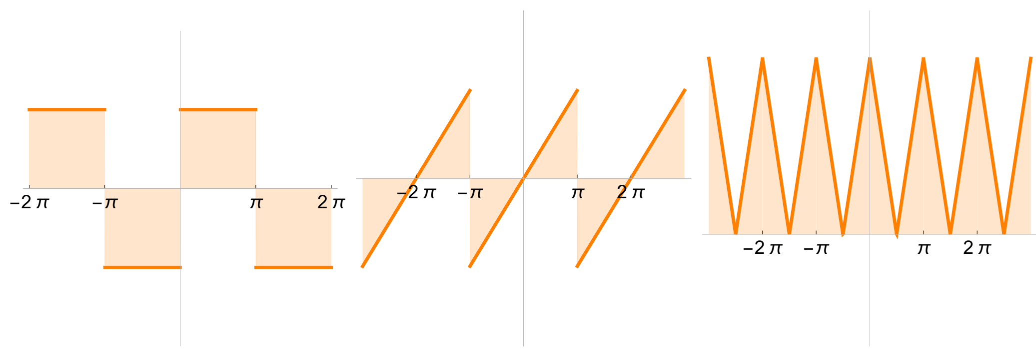

FIGURE 1. Piecewise smooth 2π-periodic functions in the linear space $\mathcal{X}$.

29.2. On the space $\mathcal{X}$ of piecewise smooth functions $f(x)$ on $[-\pi, \pi]$ there is an **inner product** defined by

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x) \, dx \; .$$

It plays the role of the **dot product** $v \cdot w$ in $\mathbb{R}^n$ or $\mathrm{tr}(A^T B)$ in $M(n,m)$.

29.3. This product allows to define angles, **length** $|f|$, **distance** $|f-g|$ or **projections** in X as we did in finite dimensions: the length is $\sqrt{\langle f, f \rangle}$. The **angle** α is defined by by $\cos(\alpha) = \langle f, g \rangle / (|f||g|)$ and the projection of f onto g is $\frac{\langle f, g \rangle}{|g|^2} g$.

29.4. The function $f(x) = x^2$ for example has length $|f| = \sqrt{(1/\pi)\int_{-\pi}^{\pi} x^4\, dx} = \sqrt{2\pi^5/(5\pi)} = \sqrt{2/5}\pi^2$. It is perpendicular to the function $g(x) = x^3$. It illustrates the general principle that even and odd functions are perpendicular to each other.

Lemma: $\{\cos(nx), \sin(nx), 1/\sqrt{2}\,\}$ form an orthonormal set in $\mathcal{X}$.

The addition formulas

$$\begin{aligned} 2\cos(nx)\cos(mx) &= \cos(nx - mx) + \cos(nx + mx) \\ 2\sin(nx)\sin(mx) &= \cos(nx - mx) - \cos(nx + mx) \\ 2\sin(nx)\cos(mx) &= \sin(nx + mx) + \sin(nx - mx) \quad . \end{aligned}$$

allow to verify these things. What helps is that integral of an odd function over $[-\pi, \pi]$ is zero:

Proof. $\langle 1/\sqrt{2}, 1/\sqrt{2}\,\rangle = 1$
$\langle \cos(nx), \cos(nx)\rangle = 1, \langle \cos(nx), \cos(mx)\rangle = 0$
$\langle \sin(nx), \sin(nx)\rangle = 1, \langle \sin(nx), \sin(mx)\rangle = 0$
$\langle \sin(nx), \cos(mx)\rangle = 0$
$\langle \sin(nx), 1/\sqrt{2}\rangle = 0$
$\langle \cos(nx), 1/\sqrt{2}\rangle = 0$ □

29.5. The **Fourier coefficients** of a function f in X are defined as

$$a_0 = \langle f, 1/\sqrt{2}\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)/\sqrt{2}\, dx$$

$$a_n = \langle f, \cos(nt)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\, dx$$

$$b_n = \langle f, \sin(nt)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\, dx$$

29.6. The **Fourier representation** of a piecewise smooth function f is the identity

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$$

We will see later that the series converges and that the identity holds at all points x where f is continuous. We first want to learn how to compute the Fourier expansion.

29.7. Remember that f is **odd** if $f(x) = -f(-x)$ for all x. A function f is **even** if $f(x) = f(-x)$ for all x.

If f is odd then f has a sin-series.
If f is even then f has a cos-series.

This follows from the fact that if the definite integral of an odd function over $[-\pi, \pi]$ is always 0, and that the product between an even and an odd function is always odd.

29.8. Find the Fourier series of $f(x) = x$ on $[-\pi, \pi]$. This is an odd function $f(-x) = -f(x)$ so that it has a sin series: with

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\ dx = \frac{-1}{\pi}(x\cos(nx)/n + \sin(nx)/n^2|_{-\pi}^{\pi}) = \frac{2(-1)^{n+1}}{n} \ ,$$

we get

$$x = \sum_{n=1}^{\infty} 2\frac{(-1)^{n+1}}{n}\sin(nx) \ .$$

29.9. If we evaluate both sides at a point x, we obtain identities. For $x = \pi/2$ for example, we get

$$\frac{\pi}{2} = 2(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}...) \ .$$

This is a **formula of Leibniz**.

29.10. Let $f(x) = 1$ on $[-\pi/2, \pi/2]$ and $f(x) = 0$ else. This is an even function $f(-x) - f(x) = 0$. It has a cos-series: with $a_0 = 1/(\sqrt{2})$, $a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1\cos(nx)\ dx = \frac{\sin(nx)}{\pi n}|_{-\pi/2}^{\pi/2} = \frac{2(-1)^m}{\pi(2m+1)}$ if $n = 2m+1$ is odd and 0 else. So, the series is

$$f(x) = \frac{1}{2} + \frac{2}{\pi}(\frac{\cos(x)}{1} - \frac{\cos(3x)}{3} + \frac{\cos(5x)}{5} - ...)$$

What happens at the discontinuity? The Fourier series converges to $1/2$. Diplomatically it has chosen the point in the middle of the limits from the right and the limit from the left. By the way, also in this example with $x = 0$, we get 1 on the left and $1/2 + 2/\pi(1 - 1/3 + 1/5 - 1/7 + \cdots)$ confirming again the Leibniz formula.

29.11. The function $f_n(x) = \frac{a_0}{\sqrt{2}} + \sum_{k=1}^{n} a_k\cos(kx) + \sum_{k=1}^{n} b_k\sin(kx)$ is called a **Fourier approximation** of f. The picture below plots a few approximations in the case of a piecewise continuous even function given in the above example.

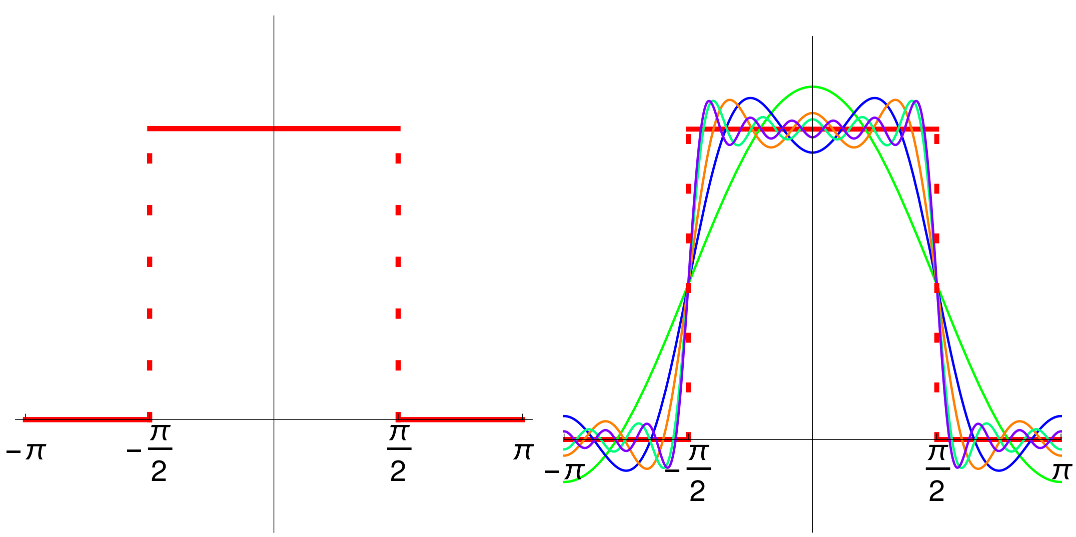

FIGURE 2. A piecewise continuous function and some Fourier approximations.

Homework

This homework is due on Tuesday, 4/16/2019.

Problem 29.1: Find the Fourier series of the function $f(x) = 22 + |6x|$.

Problem 29.2: a) Find the Fourier series of the function $4\cos(3x) + \sin^2(22x) + 22$. (Note, there is almost nothing to do here).
b) What is the Fourier series of the function which is x for $x \geq 0$ and 0 else?

Problem 29.3: a) Find the angle between the functions $f(x) = x^2$ and $g(x) = x^3$.
b) Project $f(x) = \sin^2(x)$ onto the plane spanned by $\sin(2x), \cos(2x)$.
c) Find the length of the function $f(x) = x^3$ in C^∞_{per}.

Problem 29.4: Find the Fourier series of the function $f(x) = |\sin(x)|$.

Problem 29.5: The **inner product** of two complex functions f, g in $C(\mathbb{T})$ is defined as

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)} g(x) \, dx .$$

a) Check that $\mathcal{B} = \{e^{inx}\}$ with $n \in \mathbb{Z}$ is an orthonormal family in $C^\infty(\mathbb{T})$.
b) Write $\cos(nx)$ and $\sin(nx)$ as a linear combination of functions in $\mathcal{B}$.
c) Write e^{inx} as a linear combination of real trig functions.
d) Now write $f(x) = x$, the example done in this text again, but as a Fourier series using the complex basis $\mathcal{B}$.

P.S. Fourier theory in the complex are a bit more natural. We do not have to treat three different cases like constant function, even or odd trig functions. The Fourier basis is $\{e^{inx}, n \in \mathbb{Z}\}$ which is the eigenbasis of D on the circle. The complex Fourier coefficients are

$$c_n = \langle e^{inx}, f \rangle .$$

The **complex Fourier series** is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} .$$

Oliver Knill, knill@math.harvard.edu, Math 22b, Harvard College, Spring 2019