LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Unit 28: Checklist for Second Hourly

Definitions

- **Characteristic polynomial of A** $p(\lambda) = \det(A - \lambda)$
- **Orthonormal eigenbasis** An eigenbasis which is orthonormal
- **Orthogonal complement in $\mathbf{R}^n$** space $V^\perp = \{v \in \mathbf{R}^n \mid v$ perpendicular to $V\}$
- **Transpose matrix** $A^T_{ij} = A_{ji}$. Transposition switches rows and columns.
- **Symmetric matrix** $A^T = A$ and **skew-symmetric** $A^T = -A$, are both normal
- **Structure of p:** $p(\lambda) = (-\lambda)^n + \text{tr}(A)(-\lambda)^{n-1} + \cdots + \det(A)$
- **Eigenvalues and eigenvectors** $Av = \lambda v, v \neq 0$.
- **Factorization. Have $p(\lambda) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$ with roots λ_i.**
- **Algebraic multiplicity** is k, if eigenvalue appears exactly k times.
- **Geometric multiplicity** the dimension of the kernel of $A - \lambda I_n$
- **Simple spectrum** The property of having all eigenvalues to be different.
- **Kernel and eigenspace** $\ker(A - \lambda)$ is eigenspace.
- **Eigenbasis** Basis which consists of eigenvectors of A
- **Complex numbers** $z = x + iy = |z| \exp(i\,\theta) = r\cos(\theta) + ir\sin(\theta)$
- **n-Root of 1** $e^{2\pi ik/n}$, $k = 0, \dots, n-1$ solve $\lambda^n = 1$ (are on regular n-gon)
- **Discrete dynamical system** $\vec{v}(t+1) = A\vec{v}(t)$ with initial condition $v(0)$ known
- **Continuous dynamical system** $\vec{v}'(t) = A\vec{v}(t)$ with initial condition $v(0)$ known
- **Asymptotic stability** $A^n\vec{x} \to 0$ for all $\vec{x}$ (equivalent $|\lambda_i| < 0$)
- **Markov matrix** Non-negative entries and the sum in each column is 1.
- **Mother of all ODEs** $f' = \lambda f$ has the solution $f(t) = f(0)e^{\lambda t}$.
- **Father of all ODEs** $f'' + c^2 f = 0$ has the solution $f(t) = f(0)\cos(ct) + f'(0)\sin(ct)/c$.
- **Linear operator** $T : X \to X$, where X is a linear space like a function space.

Theorems:

- **Wiggle theorem** Perturbations { symmetric matrices} can give simple spectrum.
- **Diagonalization** Matrices with simple spectrum can be diagonalized.
- **Spectral theorem** A symmetric matrix can be diagonalized with orthogonal S.
- **Spectral theorem** A normal matrix can be diagonalized with unitary S.
- **Jordan normal form theorem** Any matrix can be brought into Jordan normal form
- **Fundamental Theorem of Algebra** $p(\lambda)$ has exactly n roots
- **Asymptotic stability for discrete** $\Leftrightarrow |\lambda_i| < 1$ for all i.
- **Asymptotic stability for continuous** $\Leftrightarrow \text{Re}(\lambda_k) < 0$ for all k.

Number of eigenvalues A $n \times n$ matrix has exactly n eigenvalues.
Inequality The geometric multiplicity of an eigenvalue is $\leq$ algebraic multiplicity.
Perron-Frobenius Markov matrices have a simple eigenvalue 1.
Markov processes The product of Markov matrices is Markov.
Solution space The ODE $p(D)f = g$ has a $\deg(p)$ dimensional solution space.

Properties:

Circular matrix $Q = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$ has eigenvalues $\lambda_k = e^{2\pi ik/5}$ with $k = 0, 1, 2, \dots, 4$ and eigenvectors $v_k = [1, \lambda_k, \lambda_k^2, \lambda_k^3, \lambda_k^4]^T$.
Diagonalization S has eigenvectors of A in columns, $B = S^{-1}AS$ is diagonal
Nontrivial kernel $\Leftrightarrow \det(A) = 0 \Leftrightarrow$ have eigenvalue 0
Orthogonal Matrices A have eigenvalues of length 1.
Determinant is Product of eigenvalues. $\det(A) = \lambda_1 \cdot \dots \cdot \lambda_n$
Trace is Sum of eigenvalues. $\mathrm{tr}(A) = \lambda_1 + \dots + \lambda_n$
Geometric Multiplicity of $\lambda \leq$ Algebraic Multiplicity of λ
All different Eigenvalues $\Rightarrow$ can diagonalize.
Symmetric matrix Have real eigenvalues, can diagonalize with orthogonal S!
Eigenvalues of A^T agree with eigenvalues of A (same $p(\lambda)$)
Rank of A^T is equal to the rank of A
Reflection at k-dimensional subspace of $\mathbf{R^n}$ has eigenvalues 1 or -1.
Euler $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. Is real for $\theta = k\pi$.
De Moivre $z^n = \exp(in\theta) = \cos(n\theta) + i\sin(n\theta) = (\cos(\theta) + i\sin(\theta))^n$
Number of eigenvalues A $n \times n$ matrix has exactly n eigenvalues
Power A^k has eigenvalue λ^k if A has eigenvalue λ. Same eigenvector!
Eigenvalues of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are $\lambda_\pm = \mathrm{tr}(A)/2 \pm \sqrt{(\mathrm{tr}(A)/2)^2 - \det(A)}$
Eigenvectors of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $c \neq 0$ are $v_\pm = \begin{bmatrix} \lambda_\pm - d, c \end{bmatrix}^T$.
Rotation-Dilation Matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, eigenvalues $a \pm ib$
Rotation-Dilation Matrix has eigenvectors $[\pm i, 1]^T$
Permutation Matrix $Ae_i = e_{i+1}$, roots are on unit circle on regular polygon
Rotation-Dilation Matrix linear stable origin if and only if $|\det(A)| < 1$.
Similarity A, B with simple eigenvalues are similar if eigenvalues are the same.
Similarity A, B both symmetric are similar if eigenvalues are the same
Similarity If A, B are similar, then eigenvalues, trace and det are the same
Similarity If some λ_j of A^k has different geometric multiplicity, then not similar
Spectral Theorem Symmetric and normal matrices are diagonalizable.
Simple spectrum Matrices with simple spectrum are diagonalizable.
Cyclic matrices $Ae_i = e_{i+1}$ and $Ae_n = e_1$ has eigenvalues λ satisfying $\lambda^n = 1$.

- **Sum of cyclic matrices** Can determine all eigenvalues and eigenvectors.
- **Row sum** If every row sum is constant λ, then λ is an eigenvalue.

Algorithms:

- **Computing Eigenvalues of A** (factor $p(\lambda)$, trace, non-invertibility)
- **Use eigenvalues** to compute determinant of a matrix.
- **Computing Eigenvectors of** A (determining kernel of $A - \lambda$)
- **Calculate with complex numbers** (add, multiply, divide, take $n-$'th roots)
- **Computing algebraic and geometric multiplicities**
- **Diagonalize Matrix** (find the eigen system to produce S)
- **Decide about similarity** (e.g. by diagonalization, geometric multiplicities)
- **Solve discrete systems** (use eigenbasis to get closed-form solution)
- **Solve continuous systems** (use eigen basis to get closed-form solution)
- **Find orthonormal eigenbasis** (always possible if $A = A^T$)
- **Important ODE's** Solve $(D - \lambda)f = 0$ or $(D^2 + c^2)f = 0$. Ask Ma or Pa.

Proof seminar:

- **Rising sea**: A picture of Grothendieck to crack a nut.
- **Chaos** $T : X \to X$ has entropy limsup of $\int_X (1/n) \log(|dT^n(x)|) dx$.
- **Matrix forest theorem**. $\det(1 + L)$ is the number of rooted forests.
- **Formula of Binet** $F_{n+1}/F_n \to \phi = (\sqrt{5} + 1)/2$.
- **Cookbook** Solve homogeneous system, find particular solution, then add.
- **Cat map** on torus $T(x, y) = (2x + y, x + y)$.
- **Standard map** on torus $T(x, y) = (2x - y + c \sin(x), y)$.

People:

- **Grothendieck** (rising sea)
- **von Neumann** (wiggle)
- **Wigner** (wiggle)
- **Jordan** (Jordan normal form)
- **Kac** (Can one hear a drum?)
- **Gordon-Webb-Wolpert** (No one can't!)
- **Leonardo Pisano** (Fibonacci Rabbits)
- **Lyapunov** (Lyapunov exponent)
- **Arnold** (cat map)
- **Cayley and Hamilton** (theorem)
- **Lorentz** (Lorentz system)