LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Unit 13: Checklist for First Hourly

Definitions

- [ ] **Linear space X**: $0 \in X$, $x + y \in X$, $\lambda x \in X$
- [ ] **Linear transformation**: $T : x \mapsto Ax$, $T(x + y) = T(x) + T(y), T(\lambda x) = \lambda T(x)$.
- [ ] **Column vectors of** A: images of standard basis vectors $e_1, \dots, e_n$.
- [ ] **Coefficient matrix of the system** $Ax = b$: is the matrix A.
- [ ] **Matrix multiplication**: $[AB]_{ij} = \sum_k A_{ik}B_{kj}$
- [ ] **Row reduction steps**: Swapping rows, Scaling row, Subtracting row.
- [ ] **Row reduced**: nonzero rows have **leading** 1, columns with leading 1 are clean, every row above row with leading 1 has leading 1 to left.
- [ ] **Pivot column**: column with leading 1 in rref(A).
- [ ] **Free variable**: a variable for which we have no leading 1 in rref(A).
- [ ] **Redundant column**: column with no leading 1 in rref(A).
- [ ] **Rank of matrix** A: number of leading 1 in rref(A). Is equal dim(im(A)).
- [ ] **Nullity of matrix** A: the number of free variables. It is dim(ker(A)).
- [ ] **Kernel of matrix**: $\{x \in \mathbf{R}^m, Ax = 0\}$.
- [ ] **Image of matrix**: $\{Ax, x \in \mathbf{R}^m\}$.
- [ ] **Inverse matrix of** A: matrix $B = A^{-1}$ satisfies $AB = BA = 1$
- [ ] **Orthogonality**: dot product $v \cdot w = v^T w$ is zero. (In stats: uncorrelated if centered)
- [ ] **Length of a vector**: $\sqrt{v \cdot v}$ (=standard deviation if centered)
- [ ] **Angle between vectors**: $\cos(\alpha) = v \cdot w/(|v||w|)$ is in $[0, \pi]$ (in stats: correlation)
- [ ] **Gram-Schmidt Orthogonalization** (QR decomposition)
- [ ] **Computing Determinants** (patterns,Laplace,row reduce,partition)
- [ ] **Least square**: Produce the least square solution of a linear equation
- [ ] **Solve data fitting problem** (linear system, least square problem)
- [ ] **Algebra of matrices** (multiplication, inverse, transpose)
- [ ] $\mathcal{B} = \{v_1, \dots, v_n\}$ **spans** V: every $x \in V$ can be written as $x = a_1v_1 + \cdots + a_nv_n$.
- [ ] $\mathcal{B}$ **linear independent** V: $a_1v_1 + \cdots a_nv_n = 0$ implies $a_1 = \cdots = a_n = 0$.
- [ ] $\mathcal{B}$ **basis in** V: linear independent in V and span V.
- [ ] **Dimension of linear space** V: number of basis elements of a basis in V.
- [ ] S**-matrix** : The matrix which contains the basis vectors as columns.
- [ ] $\mathcal{B}$ **coordinates**: $c = S^{-1}v$, where $S = [v_1, \dots, v_n]$ contains $\mathcal{B}$ as columns.
- [ ] $\mathcal{B}$ **matrix**: of T in basis $\mathcal{B}$. The matrix is $B = S^{-1}AS$.
- [ ] A **similar to** B: defined as $B = S^{-1}AS$. Write $A \sim B$.
- [ ] **Orthogonal vectors** $v \cdot w = 0$

- **Length** $||v|| = \sqrt{v} \cdot v$, **Unit vector** v with $||v|| = \sqrt{v \cdot v} = 1$
- **Orthogonal set** $v_1, \dots, v_n$: pairwise orthogonal
- **Orthonormal set** orthogonal and length 1
- **Orthonormal basis** A basis which is orthonormal
- **Orthogonal to V** v is orthogonal to V if $v \cdot x = 0$ for all $x \in V$
- **Orthogonal complement** $V^\perp = \{v \in \mathbf{R}^n \mid v \text{ perpendicular to } V\}$
- **Complex numbers** $z = a + ib = re^{i\theta} = r\cos(\theta) + ir\sin(\theta)$, $\overline{z} = a - ib$, $|z|^2 = a^2 + b^2$.
- **Quaternions** $z = a + ib + jc + kd$, $\overline{z} = a - ib - jc - kd$. $N(z) = |z|^2 = a^2 + b^2 + c^2 + d^2$.

Theorems:

- **Matrix space** $M(n, m)$ is a linear space, $M(n, n)$ is an algebra.
- **Kernel and Image** $Ker(A)$ and $Im(A)$ are linear spaces.
- **Column picture** The k'th column of A is equal to Ae_k.
- **Invertibility** If T is linear invertible, then T^{-1} is linear.
- **linear independence** $\mathcal{B}$ linear independent $\Leftrightarrow$ S has trivial kernel
- **kernel and image** $\ker(A^T) = \mathrm{im}(A)^\perp$
- **Existence of dimension** Number of basis elements is the same for every basis of V.
- **kernel of** $A^T A$ $\ker(A^T A) = \ker(A)$
- **Unique QR** QR decomposition is unique
- **Rank-nullity theorem**: $\dim(\ker(A)) + \dim(\mathrm{im}(A)) = n$, where A is $m \times n$ matrix.
- **Angles and Length**: Linear trafo T is orthogonal $\Leftrightarrow$ T preserves length/angles.
- **Laplace expansion**: $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(M(i, j))$.

Properties:

- **Cute projection onto V** $P = QQ^T$ if Q has orthonormal columns
- **Gram-Schmidt** $w_i = v_i - \mathrm{proj}_{V_{i-1}} v_i$, $u_i = w_i / ||w_i||$
- **QR-factorization** $Q = [u_1 \cdots u_n]$, $R_{ii} = |w_i|$, $R_{ij} = u_i \cdot v_j$, $j > i$
- **Transpose matrix** $A^T_{ij} = A_{ji}$. Transposition switches rows and columns.
- **Symmetric matrix** $A^T = A$ and **skew-symmetric** $A^T = -A$
- **Orthogonal matrix** $A^T A = 1$ (examples are rotations or reflections)
- **Orthogonal projection** onto V is $P = A(A^T A)^{-1} A^T$
- **Least square solution** of $Ax = b$ is $x^* = (A^T A)^{-1} A^T b$
- **Determinant** $\det(A) = \sum_\pi \mathrm{sign}(\pi) A_{1\pi(1)} A_{2\pi(2)} \cdots A_{n\pi(n)}$
- **Trace** is $\mathrm{tr}(A) = \sum_i A_{ii}$, sum of diagonal elements.

- **Orthogonal projection** $P = A(A^T A)^{-1} A^T$ onto $\mathrm{im}(A)$
- **Orthogonal projections** Only 1 is both projection and orthogonal matrix
- **Kernel of** A **and** $A^T A$ are the same: $\ker(A) = \ker(A^T A)$
- **Determinant** $\det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = ad - bc$.
- **Det** $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - fha - bdi$. Sarrus rule.

Pattern: Determinant is sum of patterns, sign depends on number of up-crossings.
Determinant of triangular matrix product of diagonal entries
Det of partitioned matrix $\det(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}) = \det(A)\det(B)$.
Product properties $\det(AB) = \det(A)\det(B)$, $\det(A^{-1}) = 1/\det(A)$
Determinants $\det(SAS^{-1}) = \det(A)$, $\det(A^T) = \det(A)$.
Adding rows or columns $\det([AvB]) + \det([AwB]) = \det([A(v+w)B])$
Scaling rows or columns $\det([A(\lambda v)B]) = \lambda\det([AvB])$
Swapping of two rows $\det(B) = -\det(A)$
Adding row to given row $\det(B) = \det(A)$
rref $\det(A) = (-1)^s(\lambda_1\lambda_2\cdots\lambda_k)\det(\text{rref}(A))$, λ_i scales and s swaps
Properties of transpose $(A^T)^T = A$, $(AB)^T = B^TA^T$, $A^T + B^T = (A+B)^T$
Properties of transpose $(A^{-1})^T = (A^T)^{-1}$, $\det(A^T) = \det(A)$
Orthogonal Matrices A have $\det(A) = \pm 1$
Rotations satisfy $\det(A) = 1$ in all dimensions
QR Decomposition $A = QR$ orthogonal A, upper triangular R
QR Decomposition $|\det(A)| = R_{11}R_{22}\cdots R_{nn}$ if $A = QR$

Algorithms:

Determinants Laplace, row reduce, partition triangular, ..., patterns you can use.
Gram-Schmidt Perform Gram Schmidt
Check linear space: both for matrix space or function space
Check linear transformation: is a given transformation linear or not?
Kernel - Image : find a basis for the kernel and basis for image by row reduction.
Row reduction: scale rows, swap rows, subtract row from other row.
Matrix algebra: multiply, invert, manipulate matrices. Solve simple matrix equations
Find space orthogonal to V: write basis as rows of a matrix and find kernel.
Find transformation: use good basis, form S, B and then find $A = SBS^{-1}$.
Similarity: If A is similar to B, then $B^n = S^{-1}A^nS^n$. If A is invertible, so is B.
Find angle: $\cos(\alpha) = (v \cdot w)/(|v||w|)$. Is zero for orthogonal vectors.
Number of basis elements: is independent of basis. Allows to define dimension.
Basis of image of A: pivot columns of A form a basis of the image of A.
Basis of kernel of A: introduce free variables for each redundant column of A.
Inverse of 2×2 matrix: switch diagonal, sign the wings and divide by determinant.
Inverse of $n \times n$ matrix: Row reduce $[A|1]$ to get $[1|A^{-1}]$.
Matrix algebra: $(AB)^{-1} = B^{-1}A^{-1}$, $A(B+C) = AB + AC$, etc. no commutativity
A is invertible: $\Leftrightarrow \text{rref}(A) = 1 \Leftrightarrow$ columns are basis $\Leftrightarrow \text{rank}(A) = n, \Leftrightarrow \text{nullity}(A) = 0$
.

Transformations:

Identity: $x \mapsto x$, $1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Rotation in plane: $x \mapsto Ax$, $A = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$, counterclockwise.

Dilation in plane: $x \mapsto \lambda x$, also called scaling. Given by diagonal $A = \lambda 1$

Rotation-Dilation: $x \mapsto Ax$, $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Scale by $\sqrt{a^2+b^2}$, rotate by α

Reflection-Dilation: $x \mapsto Ax$, $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$. Scaling factor $\sqrt{a^2+b^2}$.

Horizontal and vertical shear: $x \mapsto Ax$, $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, $x \mapsto Ax$, $A = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$.

Reflection about line: $x \mapsto Ax$, $A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$.

Projection: $x \mapsto Ax$, $A = \begin{bmatrix} a^2 & ab \\ ba & b^2 \end{bmatrix}$.

Magic bullet: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ satisfies $A^2 = 0$. $\mathrm{Ker}(A) = \mathrm{Im}(A)$.

Proofs:

Axioms: set of rules which are assumed and not proven.

Monoid: an algebraic structure with associativity and neutral element.

Group: monoid where each element has an inverse.

Examples: an example can be a counter example or a prototype.

$SL(n)$: special linear group (matrices with determinant 1).

$O(n)$: orthogonal group (determinant 1 or -1). Defined by $A^T A = 1$

$SO(n)$: special orthogonal group (determinant 1).

$U(n)$: unitary group. Defined by $\overline{A}^T A = 1$

$SU(n)$: special unitary group (determinant 1).

$SU(2)$: unitary group $A = \begin{bmatrix} z & -\overline{w} \\ w & \overline{z} \end{bmatrix}$.

Inverse: $A^{-1}_{ij} = (-1)^{i+j}\det(M(j,i))/\det(\mathrm{A})$.

Landau O: $f(x) = O(g(x))$ if $|f(x)| \leq C|g(x)|$ for some constant C and all x large.

People:

Feynman (examples)

Peano (Peano axioms)

Euclid (Axioms of geometry)

Gram (QR)

Schmidt (QR)

Gauss (row reduction)

Jordan (row reduction)

Hamilton (quaternions)

Lagrange (four square theorem)

Glashow, Salam, Weinberg (electoweak unification)

Landau (complexity)