LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Unit 8: The orthogonal group

Lecture

8.1. The **transpose** of a matrix A is $A^T_{ij} = A_{ji}$, the matrix A in which rows and columns are interchanged. The **transpose operation** $A \to A^T$ is a linear map from $M(n,m)$ to $M(m,n)$. Here are some properties of this operation:

a) $(AB)^T = B^T A^T$
b) $x \cdot Ay = A^T x \cdot y$.
c) $(A^T)^T = A$
d) $(A^T)^{-1} = (A^{-1})^T$
e) $(A+B)^T = A^T + B^T$

Proof: a) $(AB)^T_{kl} = (AB)_{lk} = \sum_i A_{li} B_{ik} = \sum_i B^T_{ki} A^T_{il} = (B^T A^T)_{kl}$.
b) $x \cdot Ay = x^T Ay = (A^T x)^T y = A^T x \cdot y$.
c) $((A^T)^T)_{ij} = (A^T)_{ji} = A_{ij}$.
d) $1_n = 1_n^T = (AA^{-1})^T = (A^{-1})^T A^T$ using a).
e) $(A+B)^T_{ij} = (A+B)_{ji} = A_{ji} + B_{ji} = A^T_{ij} + B^T_{ij}$.

8.2. An $n \times n$ matrix A is called **orthogonal** if $A^T A = 1$. The identity $A^T A = 1$ encodes the information that the columns of A are all perpendicular to each other and have length 1. In other words, the columns of A form an **orthonormal basis**. [1]

8.3. Examples of orthogonal matrices are rotation matrices and reflection matrices. These two types are the only 2×2 matrices which are orthogonal: the first column vector has as a unit vector have the form $[\cos(t), \sin(t)]^T$. The second one, being orthogonal has then two possible directions. One is a rotation, the other is a reflection. In three dimensions, a reflection at a plane, or a reflection at a line or a rotation about an axis are orthogonal transformations. For 4×4 matrices, there are already transformations which are neither rotations nor reflections.

8.4. Here are some properties of orthogonal matrices:

[1] Why not call it **orthonormal matrix**? It would make sense, but **orthogonal matrix** is already strongly entrenched terminology.

a) If A is orthogonal, $A^{-1} = A^T$.
b) If A is orthogonal, then not only $A^T A = 1$ but also $AA^T = 1$.

Theorem: A transformation is orthogonal if and only if it preserves length and angle.

Proof. Let us first show that an orthogonal transformation preserves length and angles. So, let us assume that $A^T A = 1$ first. Now, using the properties of the transpose as well as the definition $A^T A = 1$, we get $|Ax|^2 = Ax \cdot Ax = A^T Ax \cdot x = 1x \cdot x = x \cdot x = |x|^2$ for all vectors x. Let α be the angle between x and y and let β denote the angle between Ax and Ay and α the angle between x and y. Using $Ax \cdot Ay = x \cdot y$ again, we get $|Ax||Ay|\cos(\beta) = Ax \cdot Ay = x \cdot y = |x||y|\cos(\alpha)$. Because $|Ax| = |x|, |Ay| = |y|$, this means $\cos(\alpha) = \cos(\beta)$. As we have defined the angle between two vectors to be a number in $[0, \pi]$ and cos is monotone on this interval, it follows that $\alpha = \beta$.
To the converse: if A preserves angles and length, then $v_1 = Ae_1, \cdots v_n = Ae_n$ form an orthonormal basis. By looking at $B = A^T A$ this shows off diagonal entries of B are 0 and diagonal entries of B are 1. The matrix A is orthogonal. ☐

8.5. Orthogonal transformations form a group with multiplication:

Theorem: The composition and the inverse of two orthogonal transformations is orthogonal.

Proof. The properties of the transpose give $(AB)^T AB = B^T A^T AB = B^T B = 1$ so that AB is orthogonal if A and B are. The statement about the inverse follows from $(A^{-1})^T A^{-1} = (A^T)^{-1} A^{-1} = (AA^T)^{-1} = 1$. ☐

8.6. Orthogonal transformations are important because they are natural **symmetries**. Many coordinate transformations are orthogonal transformations. We will also see that the Fourier expansion is a type of orthogonal transformation.

Examples

8.7. Here is an orthogonal matrix, which is neither a rotation, nor a reflection. it is an example of a **partitioned matrix**, a matrix made of matrices. This is a nice way to generate larger matrices with desired properties. The matrix

$$A = \begin{bmatrix} \cos(1) & -\sin(1) & 0 & 0 \\ \sin(1) & \cos(1) & 0 & 0 \\ 0 & 0 & \cos(3) & \sin(3) \\ 0 & 0 & \sin(3) & -\sin(3) \end{bmatrix}$$

produces a rotation in the xy-plane and a reflection in the zw-plane. It is not a reflection because A^2 is not the identity. It is not a rotation either as the determinant is not 1 nor -1. We will look at determinants later.

8.8. What is the most general rotation in three dimensional space? How many parameters does it need? You can see this when making a photograph. You use two angles to place the camera direction, but then we can also turn the camera.

8.9. What is the most general rotation matrix in three dimensions? We can realize that as

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\phi) & 0 & -\sin(\phi) \\ 0 & 1 & 0 \\ \sin(\phi) & 0 & \cos(\phi) \end{bmatrix} \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

8.10. Problem: Find a rotation which rotates the earth (latitude,longitude)=(a_1, b_1) to a point with (latitude,longitude)=(a_2, b_2)? Solution: The matrix which rotates the point $(0, 0)$ to (a, b) is a composition of two rotations. The first rotation brings the point into the right latitude, the second brings the point into the right longitude. $R_{a,b} = \begin{bmatrix} \cos(b) & -\sin(b) & 0 \\ \sin(b) & \cos(b) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(a) & 0 & -\sin(a) \\ 0 & 1 & 0 \\ \sin(a) & 0 & \cos(a) \end{bmatrix}$. To bring a point (a_1, b_1) to a point (a_2, b_2), we form $A = R_{a_2,b_2} R^{-1}_{a_1,b_1}$.
With Cambridge (USA): $(a_1, b_1) = (42.366944, 288.893889)\pi/180$ and Zürich (Switzerland): $(a_2, b_2) = (47.377778, 8.551111)\pi/180$, we get the matrix

$$A = \begin{bmatrix} 0.178313 & -0.980176 & -0.0863732 \\ 0.983567 & 0.180074 & -0.0129873 \\ 0.028284 & -0.082638 & 0.996178 \end{bmatrix} .$$

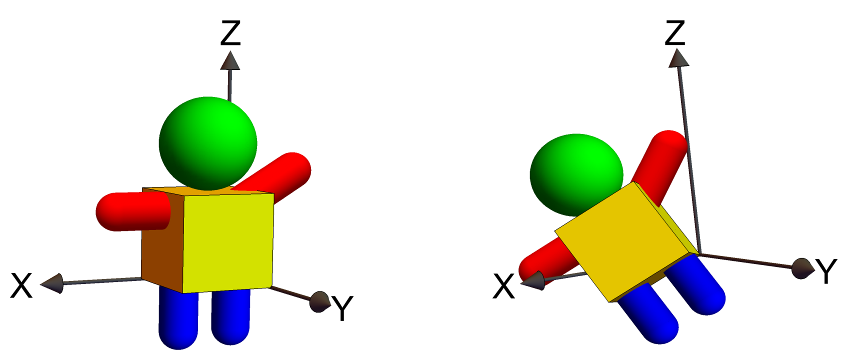

Figure 1. The rotation group $SO(3)$ is three dimensional. There are three angles which determine a general rotation matrix.

Homework

This homework is due on Tuesday, 2/20/2019.

Problem 8.1: Which matrices are orthogonal? a) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$, b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$, c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. d) $[-1]$, e) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, f) $\begin{bmatrix} \cos(7) & -\sin(7) & 0 & 0 \\ \sin(7) & \cos(7) & 0 & 0 \\ 0 & 0 & \cos(1) & \sin(1) \\ 0 & 0 & \sin(1) & -\cos(1) \end{bmatrix}$, g) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Problem 8.2: If A, B are orthogonal, then
a) Is A^T orthogonal? b) Is B^{-1} orthogonal? c) Is $A - B$ orthogonal? d) Is $A/2$ orthogonal? e) Is $B^{-1}AB$ orthogonal? f) Is BAB^T orthogonal?

Problem 8.3: Rotation-Dilation matrices $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ form an algebra in which the multiplication $(a + ib)(c + id)$ corresponds to the matrix multiplication $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$. Draw the vectors $z = \sqrt{3} + i$, $w = 1 - i$ in the complex plane, then draw its product. Find the angles α, β and lengths r,s so that $z = re^{i\alpha}, w = se^{i\beta}$ and verify that $zw = rse^{i(\alpha+\beta)}$.

Problem 8.4: Matrices of the form $A(p, q, r, s) = \begin{bmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{bmatrix}$ are called **quaternions**. a) Find a basis for the set of quaternion matrices. b) Under which condition is $A(p, q, r, s)$ orthogonal?

Problem 8.5: a) Explain why the identity matrix is the only $n \times n$ matrix that is orthogonal, upper triangular and has positive entries on the diagonal. b) Conclude, using a) that the QR factorization of an invertible $n \times n$ matrix A is unique. That is, if $A = Q_1R_1$ and $A = Q_2R_2$ are two factorizations, argue why $Q_1 = Q_2$ and $R_1 = R_2$.