LINEAR ALGEBRA AND VECTOR ANALYSIS

MATH 22B

Unit 3: Axioms

SEMINAR

3.1. An **axiom system** is a collection of statements which define a **mathematical structure** like a **linear space**. The statements of an axiom system are not proven; they are assertions which are **assumed** to be true. They need to be "interesting" in the sense that there should be realizations which satisfy these axioms. You should think of an axiom system as **the rules** of the game. There are games which are interesting like chess, there are games which are not interesting: a chess game in which the queen can jump anywhere on the board would not be interesting, as the player who starts wins the game. An axiom system stating that $x = 0$ for all x would not be interesting. Also similar to games, new interesting mathematical structures are constantly invented and played with. Playing the rules of an axiom system and finding new theorems in it is the **mathematician's game**.

3.2. In the first lecture we have seen axioms which define a **linear space**. Some linear spaces also feature a multiplicative structure and an additional set of axioms which define an **algebra**. These axioms for linear spaces are reasonable because $M(n, m)$ realizes it. The algebra structure is reasonable because $M(n, n)$ is a model for an algebra.

3.3. Here is a first example of an axiom system which is much simpler than the axiom system for a linear space. It defines the structure of a **monoid** which is an important structure in mathematics and computer science.

Definition: $(X, +, 0)$ is a **monoid**, if $+$ defines an addition on X which is associative $(x + y) + z = x + (y + z)$ and which is compatible with **zero** 0 in the sense that $x + 0 = x$ for all x in X. The addition has to be defined between any two elements in X and give again an element in X.

3.4. Examples:
a) The real numbers form a monoid with 0 as the zero element.
b) The nonzero real numbers with multiplication form a monoid. The "zero" is 1.
c) If X is a linear space, then $(X, +, 0)$ is a monoid.
d) The set of even integers defines a monoid.
e) The set of functions from $\mathbb{R}$ to $\mathbb{R}$ form a monoid with composition as addition $f \circ g$.
f) Let us define an addition of graphs $G = (V, E)$ and $H = (W, F)$, where V, W are

the vertex sets and E, F are the edge sets. The sum of the two graphs is $G + H = (V \cup W, E \cup F \cup U(V, W))$, where $U(V, W)$ is the set of all connections from V to W.
g) The set of complex numbers $\mathbb{C}$ form a monoid under addition.
h) The set of functions from $\mathbb{R}^2$ to $\mathbb{R}$ form a monoid under addition.

3.5.

Problem A: a) Verify that the natural numbers $\mathbb{N} = \{0, 1, \dots\}$ form a monoid with addition.
b) Do the negative numbers $\{-1, -2, -3, \dots\}$ form a monoid?
c) What about $X = \{0, -1, -2, -3, \dots\}$.

Problem B: a) Verify that the natural numbers $\mathbb{N} = \{0, 1, \dots\}$ form a monoid with multiplication. What is the "zero element"?
b) Verify that $M(2, 2)$ forms a monoid under multiplication. What is the zero element?

Problem C: a) Why does the set of prime numbers $(P, +)$ not form a monoid?
b) Do the numbers $\{2^n, \{n = 0, 1, 2, 3, \dots\}$ with the usual multiplication as "monoid addition" form a monoid?

3.6. Monoids play an important role also in computer science. The reason is that many **languages** are monoids. Given a finite set A called **alphabet** with letters, then a finite sequence of such letters is called a **word**. The addition of two words x, y is obtained by just **concatenating** them $x + y = xy$. For example, $x = $ "*milk*" and $y = $ "*shake*" combine to $xy = milkshake$ and $yx = shakemilk$. The alphabet could also contains punctuation signs and space, so that every text is an element in a monoid. While we usually count as a word a list of letters, in this mathematical framework, any finite sequence of letters is a word.

Problem D: a) Is the monoid of a language commutative?
b) Illustrate with an example, why the monoid of a language is associative.

3.7. To illustrate a bit more on the last example, we can make a language more interesting by giving additional rules. These additional rules are usually called a **grammar**. For example, one can look at the monoid X of words over the alphabet $A = \{a, b\}$ with the rule that b always has a letter a right before and right after. Example of words in X are *abaaaabaabaaaaaaba*.

3.8. An axiom system is called **consistent** if there is a model for it, and if one can not prove something wrong like $1 = 0$ from it. It should also not be silly like the axiom system **Null**: there is only axiom in it: **all elements are** 0**.** This axiom system obviously has a realization as the space $X = \{0\}$. It is consistent, but it is of no interest at all.

3.9. Let us look at the space $M(n,n)$ and let us define the addition $A \oplus B = AB$.

Problem E: Verify that $(M(3,3), \oplus)$ defines a monoid. What is the zero element?

3.10. An important example of a monoid is a **group**. It is a monoid, in which every element x has an inverse y. The inverse of x is an element y satisfying $x + y = 0$.

Problem F: Verify that if X is a linear space, then $(X, +, 0)$ is a group.

Problem G: Verify that the $SL(2,R)$ the subset of $M(2,2)$ for which the determinant is 1 is a group with the multiplication as addition. (Remember a formula for the inverse of a matrix.)

FIGURE 1. Euclid of Alexandria (325 BC-265 BC) built the first axiom system for geometry and Giuseppe Peano (1858-1932) formulated the Peano axioms of arithmetic.

3.11.

Homework

Exercises A)-D) are done in the seminar. This homework is due on Tuesday:

Problem 3.1 Which of the following sets form monoids? If it is, identify the zero.
a) The set of rotation dilation matrices with multiplication as addition.
b) The set of rotation dilation matrices with matrix addition as addition.
c) The set of horizontal shear matrices with multiplication as addition.
d) The set of horizontal shear matrices with matrix addition as addition.
e) The set of all sets with addition $A + B = A\Delta B$ (symmetric difference).
f) The set of all sets with addition $A + B = A \cup B$.

Problem 3.2 Which of the following sets form groups?
a) The set of all diagonal matrices with matrix addition as addition.
b) The set of all diagonal matrices with matrix multiplication as addition.
c) The set of all 2×2 matrices satisfying $A^2 = 0$.
d) The set of all 2×2 matrices for which the trace is 0 and with matrix addition as addition.
e) The set of all invertible functions from $\mathbb{R} \to \mathbb{R}$ with composition.
f) The set of all sets with addition $A + B = A\Delta B = (A \setminus B) \cup (B \setminus A)$.

Problem 3.3 Most of the axiom systems which are used in mathematics have many rules. Here is an structure, which needs only one axiom to be defined: X is a set of non-empty sets which is closed under the operation of taking finite-nonempty subsets. This structure is called a **simplicial complex**. The sets are the simplices. Let us take the example where X is the set of all non-empty subsets of $\{1, 2, 3\}$. Visualize this structure by drawing the graph in which the sets are the nodes and where two are connected, if one is contained in the other.

Problem 3.4 Look up the nine **Peano axioms** and write them down.

Problem 3.5 Look up the notions of **Magma** and **Semigroup** and compare their axiom systems with the axiom system of a monoid and group. Maybe order the structures according to their generality.