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TTH 11:30 Alison Miller

- Please fill in your name and mark your section.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) (20 points) True or False? No justifications are needed.

1) T F If A, B are similar $n \times n$ matrices, then A^5 is similar to B^5.

Solution:
$B = S^{-1}AS$, then $B^5 = S^{-1}A^5S$.

2) T F The columns of a 4×4 orthogonal matrix form an orthonormal basis of R^4.

Solution:
Yes, in that case $AA^T = 1$.

3) T F If A has orthogonal columns, then AA^T is an orthogonal projection onto the image of A.

Solution:
They need to be orthonormal.

4) T F If $\vec{v}$ is an eigenvector of a 3×3 orthogonal matrix A, then it is also an eigenvector of A^T.

Solution:
Orthogonality implies $A^T = A^{-1}$ so that A and A^T have the same eigenvectors.

5) T F If A is similar to B and A is orthogonal, then B is orthogonal.

Solution:
Conjugate an orthogonal matrix and it becomes no more orthogonal in general

6) T F The determinant of a 2×2 projection matrix onto a line is always equal to 0.

Solution:
The transformation is not invertible.

7) T F If A, B are similar 5×5 matrices, then A and B have the same nullity.

Solution:
The dimension of the kernel is the same. Use the dimension formula.

8) T F If a vertical shear has an eigenvalue 1 of algebraic multiplicity 2 then it is the identity.

Solution:
A shear always has an eigenvalue 1 with algebraic multiplicity 2. The statement would become true if we replaced algebraic multiplicity with "geometric multiplicity".

9) T F If a matrix A has the QR decomposition $A = QR$ and Q is similar to R, then Q has all eigenvalues 1.

Solution:
Since R has positive diagonal entries, all eigenvalues of R are positive. Since Q is similar to R, the eigenvalues of Q are the same. They are real and positive. An orthogonal matrix has only real eigenvalues 1 or -1. The positivity excludes -1.

10) T F For any two 2×2 matrices A, B, the characteristic polynomials of AB and BA are the same.

Solution:
The trace and determinant are the same.

11) T F Every 2×2 matrix with eigenvalues 1 and 2 is similar to a diagonal matrix.

Solution:
It is diagonalizable.

12) T F If the eigenvalues of a matrix A are the same as the eigenvalues of A^{-1}, then A is diagonalizable.

Solution:
The shear is a counter example.

13) T F If $f_A(\lambda)$ is the characteristic polynomial of a $n \times n$ matrix A, then there is a $2n \times 2n$ matrix B such that $f_A^2(\lambda)$ is the characteristic polynomial $f_B(\lambda)$ of B.

Solution:
Look at partitioned matrix with A in the diagonal.

14) T F $\det(A + 2B) = \det(A) + 2\det(B)$ is true for any two 5×5 matrices A, B.

Solution:
Take identity matrices $A = B = 1_n$.

15) T F If the projection onto the column space of a matrix A is AA^T, then the columns are an orthonormal basis in the image of A.

Solution:
It implies $A^T A = I_n$ meaning that the columns are orthonormal.

16) T F If A and B are 2×2 matrices with the same determinant and the same trace, then A and B are similar.

Solution:
The identity matrix and the shear are a counter example.

17) T F If A is the matrix of an orthogonal projection from R^3 onto a plane, then $\det A = 0$.

Solution:
It is not invertible.

18) T F If A is a 3×3 matrix then $(A - 5I_3)^T$ has the same eigenvalues as $(A - 5I_3)$.

Solution:
The matrix $B = A - 5I_3$ has the same eigenvalues than B^T.

19) T F The **regular transition** matrix $\begin{bmatrix} 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 0 \\ 0 & 1/3 & 3/4 \end{bmatrix}$ has an eigenvalue of 1.

Solution:
Its transpose has row vectors for which all entries sum up to 1.

20) | T | F | If two matrices A, B have the same eigenvalues with the same corresponding algebraic multiplicities, then A and B are similar.

Solution:
The shear and the identity matrix are counter examples.

Problem 2) (10 points)

a) (4 points) No justifications are necessary for this problem. Which of the following matrices are diagonalizable over the complex numbers?

1) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ ☐ 2) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 4 \end{bmatrix}$ ☐

3) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ ☐ 4) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ ☐

b) (6 points) No justifications are necessary. Which matrices are orthogonal, which have an eigenvalue -1?

Matrix A	A is orthogonal	A has an eigenvalue -1
$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	☐	☐
$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	☐	☐
$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$	☐	☐

Solution:
a) 2),3),4) are true.
b) First row: True True
Second row: True False
Third row: False False

Problem 3) (10 points)

A **Walsh matrix** is a square matrix with dimension 2^n with ± 1 entries and orthogonal columns.

Joseph Walsh was Harvard Professor from 1935 to 1966. His picture hangs in the Math common room.

a) (3 points) What is the determinant and the QR decomposition of the 2×2 Walsh matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} .$$

b) (4 points) Find the QR decomposition of the 4×4 Walsh matrix

$$B = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix} .$$

c) (3 points) Find the determinants of both A and B.

Solution:
a) $det(A) = -2$, $A = R/\sqrt{2} \cdot (\sqrt{2} I_2)$ is the QR decomposition.
b) $QR = (B/2) \cdot (2 \cdot I_2)$.
c) $detB = det(Q) \cdot det(R) = (-1) \cdot (2^4)$, which gives the answer up to a sign. In order to see the sign of Q (or compute the determinant directly) row reduce A. Subtract the first from the second and the third from the last to get

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 0 & -2 & 2 \\ 1 & -1 & -1 & -1 \\ 0 & 0 & 2 & 2 \end{bmatrix} .$$

Now switch the 3. and 4. row to get a partitioned matrix which has determinant $(-2)(-8) = 16$.

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix} .$$

Since we have switched once rows, the determinant is $\boxed{-16}$.

Problem 4) (10 points)

Define the **Harvard matrix** $H = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$.

a) (2 points) Verify that $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ are eigenvectors of H.

b) (4 points) Find all eigenvalues, eigenvectors and the geometric multiplicities of the eigenvalues of H.

c) (2 points) Is H diagonalizable? If yes, write down the diagonal matrix B such that

$$B = S^{-1}HS \; .$$

d) (2 points) Find the characteristic polynomial $f_H(\lambda)$ of H.

Solution:
a) Just compute Av and compare with v to see the eigenvalues. They are 2 and 4.
a) The matrix has a 4 dimensional kernel as you can see if you row reduce the matrix. The eigenvectors to the eigenvalue 0 can be read off after doing row reduction of A:

$$rref(A) = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

There are 4 eigenvalues 0. which is the sum of the eigenvalues. Here are some eigenvectors to the eigenvalues 0 are

$$\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

c) Since the matrix has an eigenbasis, it is diagonalizable and similar to the diagonal matrix which contains the eigenvalues in the diagonal:

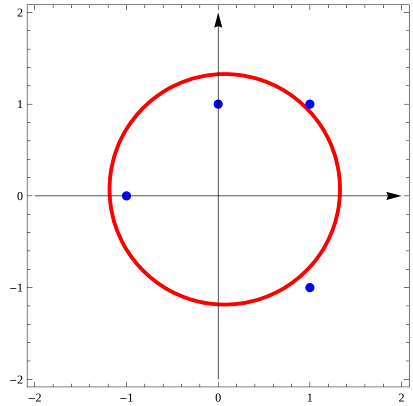
$$B = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

c) We know all the eigenvalues λ_i. The characteristic polynomial is $f_A(\lambda) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda)$. Therefore, $f_A(\lambda) = \boxed{(-\lambda)^4(2-\lambda)(4-\lambda)}$.

Problem 5) (10 points)

Find the circle $a(x^2 + y^2) + b(x + y) = 1$ which best fits the data

x	y
0	1
-1	0
1	-1
1	1



In other words, find the least square solution for the system of equations for the unknowns a, b which aims to have all 4 data points (x_i, y_i) on the circle.

Solution:
To get system of linear equations $Ax = b$, plug in the data

$$\begin{aligned} lla + b &= 1 \\ a - b &= 1 \\ 2a &= 1 \\ 2a + 2b &= 1 \,. \end{aligned}$$

This can be written as $Ax = b$, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \\ 2 & 2 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} .$$

We get the least square solution with the usual formula. First compute

$$(A^T A)^{-1} = \begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix} /22$$

and then

$$A^T b = \begin{bmatrix} 6 \\ 2 \end{bmatrix} ,$$

so that the least square solution is

$$x = (A^T A)^{-1} A^T b = \begin{bmatrix} 7/11 \\ -1/11 \end{bmatrix} .$$

The best circle is $\boxed{7(x^2 + y^2) - (x + y) = 11.}$

Problem 6) (10 points)

a) (4 points) Find all the eigenvalues λ_1, λ_2 and eigenvectors v_1, v_2 of the matrix

$$A = \begin{bmatrix} 6 & -5 \\ 1 & 0 \end{bmatrix} .$$

b) (6 points) Find a closed form solution for the recursion

$$x(n+1) = 6x(n) - 5x(n-1)$$

for which $x(0) = 3$, $x(1) = 7$.

Solution:
Since the sum of the rows is constant, one can see that 1 is an eigenvalue to the eigenvector $[1, 1]^T$. The second eigenvalue can be seen from the trace. It is 5. Its eigenvector is the kernel of $A - 5I_2$ leading to $[5, 1]^T$ as an eigenvector. Write the initial condition as a sum of eigenvectors. This system of equations $[7, 3]^T = c_1[1, 1]^T + c_2[5, 1]^T$ has the solution $c_1 = 2, c_2 = 1$. We have the closed form solution

$$A^n \begin{bmatrix} 7 \\ 3 \end{bmatrix} = 2 \cdot 1^n \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \cdot 5^n \cdot \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 + 5 \cdot 5^n \\ 2 + 5^n \end{bmatrix} .$$

Problem 7) (10 points)

a) (3 points) Find the determinant of

$$\begin{bmatrix} 0 & 0 & 7 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b) (3 points) Find the determinant of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \end{bmatrix}$$

c) (4 points) Find the determinant of the matrix

$$\begin{bmatrix} 5 & 2 & 3 & 2 & 0 & 0 \\ 6 & 2 & 3 & 2 & 0 & 0 \\ 5 & 3 & 3 & 2 & 0 & 0 \\ 5 & 2 & 3 & 3 & 0 & 0 \\ 2 & 1 & 8 & 3 & 4 & 5 \\ 3 & 6 & 9 & 0 & 3 & 4 \end{bmatrix}$$

Solution:
a) There is only one pattern which is nonzero

$$\begin{bmatrix} 0 & 0 & \boxed{7} & 0 & 0 & 0 \\ \boxed{8} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} \\ 3 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{5} & 0 \\ 0 & \boxed{2} & 0 & 0 & 0 & 0 \end{bmatrix} .$$

There are 7 upcrossings. The determinant is $(-1)^7 \cdot 7 \cdot 8 \cdot 1 \cdot 1 \cdot 5 \cdot 2 = \boxed{-560}$.
b) First change the order of rows to get a matrix which has 2's in the diagonal and 1's in the rest. After 10 row swaps, we are here:

$$B = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

Now subtract 1_5 from the diagonal to get a matrix with eigenvalues $0, 0, 0, 0, 5$ so that B has eigenvalues $1, 1, 1, 1, 6$ and determinant 6. The determinant of A is also $\boxed{6}$.
c) Partition the matrix to have a 4×4 matrix block and a 2×2 matrix block. Row reduce the top left matrix to get 3. The lower right 2x2 matrix has determinant 1. The final answer is $\boxed{3}$.

Problem 8) (10 points)

Find all (possibly complex) eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

Solution:
The matrix A is $2 * Q^2$ where

$$Q = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

We have computed the eigenvalues and eigenvectors of Q several times already: $\lambda_k = e^{2\pi i k/6}$, where $k = 0, ..., 5$. The eigenvectors of Q

$$v_k = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \lambda^3 \\ \lambda^4 \\ \lambda^5 \end{bmatrix} .$$

These eigenvectors are also the eigenvectors of A but to the eigenvalues $2\lambda_k^2 = 2e^{2\pi i 2k/6}$ which is $2e^{2\pi i k/3} = 2\cos(2\pi k/3) + i2\sin(2\pi k/3)$, where $k = 0, 1, 2, 3, 4, 5$. It turns out that eigenvalues appear with algebraic multiplicity 2. This made this problem harder to solve directly. By the way, you have solved the 4×4 case in the homework without the scaling factor.

Problem 9) (10 points)

Let $U = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$ define the 6×6 matrix $A = U \cdot U^T$.

a) (4 points) Find a basis for the image of A and find the dimension of the kernel of A.

b) (3 points) What are the eigenvalues of A with their algebraic and geometric multiplicities?

c) (3 points) Find $\det(A + 2I_6)$ and $\operatorname{tr}(A + 2I_6)$.

Solution:
a) The image has the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \right\}$$

and is one dimensional By the rank-nullity theorem The kernel has dimension 5.
b) The eigenvalues are 0 and since U is an eigenvector with eigenvalue $UU^T = 1 + 4 + 9 + 16 + 25 + 36 = 91$:

$$AU = UU^T U = (UU^T)U = 91U \ .$$

The eigenbasis for the eigenvalue 91 is $\mathcal{B}$ from a). The space perpendicular to this line is the eigenspace to the eigenvalue 0. The geometric and algebraic multiplicities of the 0 eigenvalue are 5.
c) The eigenvalues of $A + 2I_6$ are $2 + 91 = 93$ (with algebraic multiplicity 1) and 2 (with algebraic multiplicity 5). The determinant of $A + 2I_6$ is the product of the eigenvalues which is $93 * 2^5 = \boxed{93 * 32}$. The trace of $A + 2I_6$ is the sum of the eigenvalues which is $\boxed{103}$.

Problem 10) (10 points)

Let V be the 3-dimensional subspace of $\mathbb{R}^4$ spanned by

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Let $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ be an orthonormal basis of V and A be the matrix $A = \begin{bmatrix} | & | & | \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \\ | & | & | \end{bmatrix}$.

Find $\det(AA^T)$ and $\det(A^T A)$.

Hint. Think before calculating. You don't actually need to compute $\vec{w}_1, \vec{w}_2, \vec{w}_3$, nor do you have to multiply matrices to solve the problem.

Solution:
Note that neither A nor A^T are square matrices so that it does not make sense to talk about $\det(A)$ and $\det(A^T)$.
However, both A^TA and AA^T are square matrices for which the determinant is defined.
The matrix A^TA has a different shape then the matrix AA^T.
As a background for this problem, take the formula for the least square solution $\vec{x}_* = (A^TA)^{-1}A^T\vec{b}$ of $A\vec{x} = \vec{b}$ which leads to the general projection formula $P\vec{b} = A(A^TA)^{-1}A^T\vec{b}$ onto the column space of a matrix A. In the case, when the column vectors of A form an orthonormal set, then this formula simplifies to $P\vec{b} = AA^T\vec{b}$ because A^TA is then an identity matrix. We have an orthonormal set of vectors as columns of A so that A^TA is an identity matrix and AA^T is a projection from four to three dimensions and not invertible. Here are the solutions:

a) AA^T is a 4×4 matrix which has its image contained in the image of A. It is the projection from a four dimensional space to a three dimensional space and therefore not invertible. Therefore $\boxed{\det(AA^T) = 0}$.

b) A^TA is the 3×3 identity matrix and has determinant $\boxed{\det(A^TA) = 1}$.