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- Please fill in your name and mark your section.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) (20 points) True or False? No justifications are needed.

1) T F

Let A and B be two 2×2 matrices. Then A and B are similar if and only if they have the same trace and determinant.

Solution:
The shear and the identity is a counter example.

2) T F

An orthogonal $n \times n$ matrix satisfies $AA^T = I_n$.

Solution:
This follows from $A^T A = I_n$.

3) T F

The eigenvectors of a matrix A do not change under row reduction.

Solution:
Look at the shear. It has only one eigenvector. After row reduction, it has two different eigenvectors.

4) T F

If A is the matrix of a reflection at a line in space, then $\det(A^2 - I_3) = 0$.

Solution:
Indeed, every nonzero vector perpendicular to the line is an eigenvector to the eigenvalue -1, and every nonzero vector in the line is an eigenvector to the eigenvalue 1. We have therefore eigenvalues 1 or -1.

5) T F

Every rotation matrix can be diagonalized over the reals.

Solution:
There can be complex eigenvalues.

6) T F

The recursion $x_{n+1} = x_n + x_{n-1} + x_{n-2} + x_{n-3}$ can be written as $\vec{v}(n+1) = A\vec{v}_n$ for a 4×4 matrix A and a vector $\vec{v}$.

Solution:
Do it $v(n) = (x_n, x_{n-1}, x_{n-2}, x_{n-3})$.

7) T **F** The are two rotations in space such that their product is an orthogonal projection onto a plane.

Solution:
Rotations are invertible. A projection is not.

8) **T** F The nullity of a square A is the same as the nullity of A^T.

Solution:
They have the same eigenvalues and geometric multiplicity.

9) T **F** For any matrix, we have $\det(A^5/2) = \det(A)^5/2$.

Solution:
We have $\det(AB) = \det(A)\det(B)$ but not $\det(A/2) = \det(A)/2$.

10) **T** F The trace of a matrix is the sum of the eigenvalues

Solution:
It is indeed the sum of the eigenvalues.

11) T **F** There is a reflection for which the determinant is equal to 2.

Solution:
The eigenvalues are 1 or -1.

12) **T** F The matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 4 & 0 \\ 1 & 3 \end{bmatrix}$.

Solution:
Because both have the same trace and determinant, their eigenvalues are the same. Because the eigenvalues are different they can both be diagonalized to the same diagonal matrix.

13) **T** F If two 2×2 matrices A and B have the same trace, determinant and geometric multiplicities of all eigenvalues are 1 then they are similar.

Solution:
The shear A and the shear A^2 are not conjugated, but they have the same trace and determinant.

14) T F If A has no kernel, then the least square solution of $Ax = b$ is unique.

Solution:
Yes, then the quadratic error is zero.

15) T F If $A^n\vec{v}$ and $A^{-n}\vec{v}$ both stay bounded for every vector $\vec{v}$, then all eigenvalues of A are located on the unit circle $|\lambda| = 1$ in the complex plane.

Solution:
We must have λ_i^n staying bounded for all n.

16) T F If an orthogonal matrix Q is symmetric, then Q is diagonal.

Solution:
Take a reflection at a line.

17) T F If $A = QR$ is the QR decomposition of a square matrix, then the eigenvalues of A are the diagonal entries of R.

Solution:
Take the case, where A is an orthogonal matrix. Then $A = A1$ is the QR decomposition, but the eigenvalues of A are not necessarily all 1.

18) T F For every 2×2 matrix A, we have $A - I_n$ is similar to $A + I_n$.

Solution:
The eigenvalues are different.

19) T F If A is similar to B then A^{-1} is similar to B^{-1}.

Solution:
$S^{-1}AS = B$ can be inverted to get $S^{-1}A^{-1}S = B^{-1}$.

20) T F For any 2×2 regular transition matrix, the trace determines the determinant.

Solution:
There is an eigenvalue 1 so that the other eigenvalue is determined by the trace. Then, we also know the determinant.

Total

Problem 2) (10 points) No justifications are needed.

a) (6 points) Which of the following assertions are true?

true	false	
		A is similar to B
		A is similar to C
		A is similar to D

true	false	
		B is similar to C
		B is similar to D
		C is similar to D

$A = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$

Solution:
A is similar to D and B is similar to C. The other cases are not similar. We can look at the trace and determinants to determine most cases. The eigenvalues of A can be read off directly. It is 3 and -1. Since they are different, one can diagonalize this matrix. The matrix D has the same trace and determinant than A and therefore also the same eigenvalues than A. Both A and D are similar to a diagonal matrix.

b) (4 points) Match the following matrices with sets of eigenvalues. You are told that there is a unique match. It is not always necessary to compute all the eigenvalues to do so.

Eneter i),ii),iii) or iv)	The matrix
	$A = \begin{bmatrix} -1 & -2 & 8 \\ -7 & -3 & 19 \\ -3 & -2 & 10 \end{bmatrix}$
	$A = \begin{bmatrix} 5 & -9 & -7 \\ 0 & 5 & 2 \\ 0 & 0 & 6 \end{bmatrix}$
	$A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
	$A = \begin{bmatrix} 13 & 11 & 13 \\ -2 & -1 & -2 \\ -8 & -7 & -8 \end{bmatrix}$

i) $\{3, 2, 1\}$.

ii) $\{1, 0, 3\}$.

iii) $\{6, 5, 5\}$.

iv) $\{1, i, -i\}$.

Solution:
b) i), iii),iv) ii). Note that the trace of the matrix is the sum of the eigenvalues. The trace determines the match already.

Problem 3) (10 points) No justifications are needed

a) (5 points) Given a unit column vector written as a 4×1 matrix $A = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$. Fill in the values of the determinants and eigenvalues of the following matrices and state whether they are symmetric

Matrix	Determinant	Eigenvalues	Symmetric? Yes/No
$A^T A$			
AA^T			
$A(A^T A)^{-1}A^T$			

Solution:
The matrix $A^T A$ is equal to the 1×1 matrix 1. The matrix AA^T is the same as $A(A^T A)^{-1}A^T$ and is a projection onto the line.

Matrix	Determinant	Eigenvalues	Symmetric? Yes/No
$A^T A$	1	1	yes
AA^T	0	0,1	yes
$A(A^T A)^{-1}A^T$	0	0,1	yes

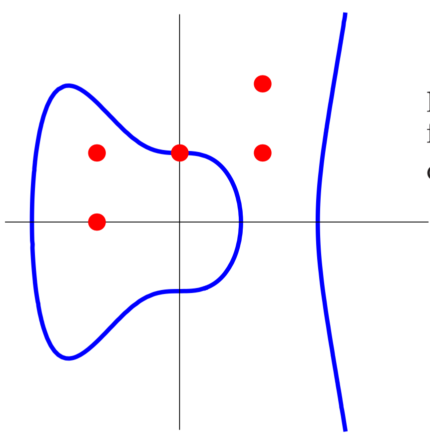
b) (5 points) No explanations are necessary for this problem.

Which matrices are orthogonal, which matrices are anti-symmetric $A^T = -A$. Which matrices are projections? Check everything which applies. It is not excluded that you have to check several properties for each matrix.

	orthogonal	antisymmetric	projection	
1)				$A = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$
2)				$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
3)				$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
4)				$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Solution:
1),2) only orthogonal
3) only projection
4) nothing

Problem 4) (10 points)



A curve of the form

$$y^2 = x^5 + ax^3 + b$$

provides examples of **hyper elliptic curves**. Use data fitting to find the best parameters (a, b) for an elliptic curve given the following points:

$$\begin{aligned} (x_1, y_1) &= (1, 2) \\ (x_2, y_2) &= (-1, 0) \\ (x_3, y_3) &= (1, 1) \\ (x_4, y_4) &= (-1, 1) \end{aligned}$$

Solution:
Setting up the equations gives the system

$$\begin{array}{rcl} a+b &=& 3 \\ -a+b &=& 1 \\ a+b &=& 0 \\ -a+b &=& 2 \end{array}$$

which can be written as $A\vec{x} = \vec{b}$, where $\vec{x} = [a,b]^T$ and $\vec{b} = [3,1,0,2]^T$. The least square solution is given by $(A^TA)^{-1}A^T\vec{b}$. One has $A^TA = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ and $(A^TA)^{-1} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/4 \end{bmatrix}/20$ and $A^T\vec{b} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$ and $(A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} 0 \\ 3/2 \end{bmatrix}$. The best fit is given by the Elliptic curve $\boxed{y^2 = 0x^3 + 3/2}$.

Problem 5) (10 points)

The sequence $2, 1, -1, -5, -13, -29, -61, \ldots$ satisfies the recursion

$$x(n+1) = 3x(n) - 2x(n-1)$$

with $x(0) = 2, x(1) = 1$. Find $x(n)$.

Hint. As usual, we write the recursion first in the form $\vec{v}(n+1) = A\vec{v}(n)$, where A is a 2×2 matrix, where $\vec{v}(n) = \begin{bmatrix} x(n+1) \\ x(n) \end{bmatrix}$ and where the initial condition is $\vec{v} = \vec{v}(0) = \begin{bmatrix} x(1) \\ x(0) \end{bmatrix}$.

Solution:
We have

$$\vec{v}(n) = \begin{bmatrix} x(n+1) \\ x(n) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(n) \\ x(n-1) \end{bmatrix} .$$

The matrix $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ has the eigenvalue 1 with eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the eigenvalue 2 with eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Since

$$\vec{v}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + - \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

we get the closed form solution

$$A^n \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 31^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + -2^n \begin{bmatrix} 2 \\ 1 \end{bmatrix} .$$

We have $\boxed{x(n) = 3 - 2^n}$.

Problem 6) (10 points)

a) (4 points) A 2×2 matrix A satisfies $\mathrm{tr}(A^2) = 5$ and $\mathrm{tr}(A) = 3$. Find $\det(A)$.

b) (3 points) A 2×2 matrix has two parallel columns and $\mathrm{tr}(A) = 5$. Find $\mathrm{tr}(A^2)$.

c) (3 points) A 2×2 matrix A has $\det(A) = 5$ and positive integer eigenvalues. What is the trace of A?

Solution:
a) $\lambda_1^2 + \lambda_2^2 = 5, \lambda_1 + \lambda_2 = 3$. Solving this for λ_1 gives $\lambda_1 = 2$ and $\lambda_2 = 1$. The determinant is $2 * 1$.
b) Since one eigenvalue is 0 and by the trace the second eigenvalue is 5, the eigenvalues of A^2 are 5^2 and 0. Therefore, $\mathrm{tr}(A^2) = 25$.
c) The only possible way that 5 is a product of two positive integers is $1 * 5$. The trace is 6.

Problem 7) (10 points)

a) (3 points) Find the determinant of the matrix

$$\begin{bmatrix} 1000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1001 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1002 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 1003 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 1004 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 & 1005 & 5 & 5 \\ 6 & 6 & 6 & 6 & 6 & 6 & 1006 & 6 \\ 7 & 7 & 7 & 7 & 7 & 7 & 7 & 1007 \end{bmatrix}$$

b) (2 points) Find the determinant of the matrix

$$\begin{bmatrix} 3 & 1 & 1 & 2 & 2 & 2 \\ 0 & 3 & 1 & 2 & 2 & 2 \\ 0 & 0 & 3 & 2 & 2 & 2 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 5 & 1 & 4 \end{bmatrix}.$$

c) (3 points) Find the determinant of the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 8 & 9 & 2 & 1 \\ 5 & 2 & 0 & 8 & 9 & 2 & 1 \\ 6 & 4 & 3 & 8 & 9 & 2 & 1 \end{bmatrix}.$$

d) (2 points) Find the determinant of the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Solution:
a) Compute the eigenvalues of A. To do so, first subtract $1000 \cdot 1_8$ from the matrix. $A - 1000 \cdot 1_8$ has the eigenvalues 0 with multiplicity 7 and 28 with multiplicity 1 (look at the trace). Therefore, A has eigenvalues 1000 with algebraic multiplicity 7 and 1028 so that the determinant is $\boxed{1000^7 \cdot 1028}$.
b) Partition the matrix to see that the determimant is $3^3 \cdot 4^3 = \boxed{12^3}$.
c) Partition and determine the sign $6 * (16 - 1) = \boxed{90}$.
d) There is only one pattern with 11 up-crossings. The determinant is $\boxed{-1}$.

Problem 8) (10 points)

a) (4 points) Find the eigenvalues of the following matrix.

$$A = \begin{bmatrix} 3 & 2 & 3 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} .$$

b) (2 points) Is there a real eigenbasis of A?

c) (4 points) Find the QR decomposition of the matrix A.

Solution:
a) Look at the two 3×3 blocks of the partitioned matrix. The eigenvalues are $3, 4, 5$ for the first matrix and $1, e^{2\pi i/3} = (1 + \sqrt{3}i)/2, e^{2\pi i 2/3} = (1 - \sqrt{3}i)/2$ for the second (this is a special shift case).
b) There is no real eigenbasis because we have complex eigenvalues.
c) Make the QR decomposition of the submatrices. The upper left matrix is already triangular, the lower right already orthogonal.

$$A = \begin{bmatrix} 3 & 2 & 3 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & 3 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 9) (10 points)

a) (3 points) Find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 0 & 7 \\ 0 & 5 & 0 \\ 2 & 0 & 9 \end{bmatrix}$.

b) (4 points) Find the eigenvectors of A.

c) (3 points) Find a matrix S such that $SB = AS$, where B is diagonal.

Solution:
a) You might spot that the matrix leaves the xz plane invariant and is there the transformation $\begin{bmatrix} 4 & 7 \\ 2 & 9 \end{bmatrix}$ with eigenvalues 11 and 2. (Look at the trace and notice that the sum of the rows add up to 11). The eigenvalues are $5, 11, 2$.
b) $\begin{bmatrix} 0, 1, 0 \end{bmatrix}$, $\begin{bmatrix} 1, 0, 1 \end{bmatrix}$, $\begin{bmatrix} 7, 0, -2 \end{bmatrix}$
c) The matrix S contains the eigenvectors as columns:

$$S = \begin{bmatrix} 0 & 1 & 7 \\ 1 & 0 & 0 \\ 7 & 1 & -2 \end{bmatrix} .$$

Because we have an eigenbasis, the matrix $B = S^{-1}AS$ is diagonal.

Problem 10) (10 points)

a) (2 points) What can you say about the absolute value of an eigenvalue of an orthogonal matrix?

b) (2 points) What can you say about the eigenvalues of an orthogonal projection?

c) (2 points) What can you say about the eigenvalues of a shear?

d) (2 points) What are the eigenvalues of a reflection at a line?

e) (2 points) You know that an eigenvalues of A is 3 and an eigenvalues of B are 5, can you conclude that an eigenvalues of AB is 10?

Solution:
a) They all have absolute value 1.
b) They are all either 1 or 0.
c) They are all 1.
d) They are all 1 or -1. e) No, we can not conclude that. Take $A = Diag(3, 0)$ and $B = Diag(0, 5)$, then AB has only the eigenvalues 0.