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- Please fill in your name and mark your section.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- All matrices are real matrices unless specified otherwise.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) True or False? No justifications are needed.

1) [T] F — The matrix $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ has the eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Solution:
Just check it. Or see that the image is one dimensional. Any eigenvector to a non-zero eigenvalue must be parallel to that direction. And there is a non-zero eigenvalue as the trace is non-zero. By the way, since 0 is an eigenvalue, the trace is the other eigenvalue.

2) T [F] — If a 2×2 matrix A has trace 2018, then the trace of $B = A - 2018I_2$ is zero.

Solution:
Take the diagonal matrix with diagonal entries $2018, 0$. The matrix B has diagonal entries $0, -2018$.

3) T [F] — If two 2×2 matrices A and B each have an eigenvalue of 2018, then the matrix $A - B$ has an eigenvalue of 0.

Solution:
Take $A = Diag(2018, 0)$ and $B = Diag(0, 2018)$. Then $A - B = Diag(2018, -2018)$ has no eigenvalue 0.

4) [T] F — If A is invertible, then $A^T A$ is invertible.

Solution:
The determinant of A is non-zero. So, the determinant of $A^T A$ is non-zero.

5) T [F] — There is an invertible matrix A such that A^T is not invertible.

Solution:
The determinant of A is the same than the determinant of A^T. The determinant DETERMINES whether A is invertible or not.

6) T [F] — Every nonzero 2×2-matrix A satisfying $A^2 = 0$ can be diagonalized.

Solution:
A counter example is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

7) T F Let A be a 1×2-matrix. If the trace $\text{tr}(A^T A) = 0$ then $A = 0$.

Solution:
Write $A = [a, b]$. Then $tr(A^T B) = a^2 + b^2$. If that is zero, then $A = 0$.

8) T F Every orthogonal projection has an eigenbasis.

Solution:
Yes, it is symmetric.

9) T F If A and B are similar, then they have the same rank.

Solution:
They have the same kernel and by the rank-nullety theorem the same rank.

10) T F Let A, B be two 2×2 rotation matrices with the same trace. Then they have the same eigenvalues.

Solution:
They both the same characteristic polynomial

11) T F The trace of a matrix A does not change under row reduction.

Solution:
Already a scaling does change the trace.

12) T F A discrete dynamical system $\vec{v}(t+1) = A\vec{v}(t)$ with 2×2 matrix A is stable if $\det(A) < 1$.

Solution:
By definition.

13) T F Let A be a 2×2 reflection dilation matrix, reflecting at a line. Then the trace of A is zero.

Solution:
One eigenvalue is 1, the other is -1. The sum is zero. One can also see it by writing down the matrix with $\cos(2\theta), \sin(2\theta)$ entries.

14) T F If A is an arbitrary $n \times n$ matrix, then $A + A^T$ is diagonalizable.

Solution:
Yes, it is symmetric

15) T F The sum of the algebraic multiplicities of a symmetric $n \times n$ matrix A can be smaller than n.

Solution:
There are n real eigenvalues.

16) T F A 2×2 matrix A such that $\text{tr}(A) = 3$, $\det(A) = 5$ is diagonalizable over the reals.

Solution:
It is diagonalizable over the complex but not over the reals.

17) T F There is a 4×4 matrix A such that $A + I$ has rank two and $A - 2I$ has rank one.

Solution:
This would imply the characteristic polynomial has at least five roots

18) T F If A is a 2×2 matrix with eigenvalues $\lambda_1 = 5$, $\lambda_2 = 5$, then $A - 5I_2$ has rank two.

Solution:
$A - 5I$ could has rank one

19) T F The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ is diagonalizable.

Solution:
It is not even square

20) | T | F |

If A, B are diagonalizable and have the same eigenvalues, then $A - B$ has all eigenvalues zero.

Solution:
Take $\text{Diag}(2, 1)$ and $\text{Diag}(1, 2)$.

| | Total |

Problem 2) (10 points) No justifications are needed.

a) (3 points) Which matrices A have the property that the discrete dynamical system

$$v(t+1) = Av(t)$$

is asymptotically stable?

Matrix A	asymptotically stable	not asymptotically stable
$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$		
$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$		
$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$		

Solution:
The first and last matrix have eigenvalues 0, the second matrix has one eigenvalue 1. The middle matrix is not asymptotically stable

b) (2 points) Which matrix A encodes the discrete dynamical system $v(t+1) = Av(t)$ if

$$v(t+1) = [x(t+1), x(t)]^T = [2x(t) + x(t-1), x(t)]^T = Av(t) \ ?$$

Check exactly one matrix

Matrix	
$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$	

Matrix	
$A = \begin{bmatrix} 2 & 1 \end{bmatrix}$	

Matrix	
$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$	

Solution:
It is the third matrix.

c) (2 points) Which of the following complex numbers are real? Remember that we defined $w^z = e^{z \log(w)}$ and $\log(z) = \ln|z| + i\,\theta$ with $0 \leq \theta < 2\pi$ for any complex numbers $w \neq 0$, $z \neq 0$, where θ is the angle so that $z = |z|e^{i\,\theta}$.

Number	is real	is not real
$i \log(i)$		
$i + i$		

Number	is real	is not real
i^2		
i^i		

Solution:
$i\log(i) = -\pi/2$ is real. $i + i$ is not real. $i^2 = -1$ is real. $i^i = e^{-\pi/2}$ is real.

d) (3 points) For each type of matrix check every box such that the corresponding property always holds for that type of matrix. By diagonalizable, we mean "diagonalizable with all eigenvalues being real".

	invertible	diagonalizable	symmetric	real eigenvalues
Projection matrix				
Shear matrix				
Rotation matrix				
Reflection matrix				
$A^2 = 0, A \neq 0$				
Diagonal matrix				

Solution:

	invertible	diagonalizable	symmetric	real eigenvalues
Projection matrix		x	x	x
Shear matrix	x			x
Rotation matrix	x			
Reflection matrix	x	x	x	x
$A^2 = 0, A \neq 0$				x
Diagonal matrix		x	x	x

Problem 3) (10 points) No justifications are needed

a) (3 points) The following 6 matrices can be grouped into 3 pairs of similar transformations. Find these three pairs.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}, E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solution:
A and C both have eigenvalues $1, 0, 3$ so that they both are similar to the same diagonal matrix. B and D are transpose of each other. They are similar. E and F both have geometric multiplicity 2 for the eigenvalues $\lambda_1 = 1, \lambda_2 = 1$ and an other common eigenvalue $\lambda_3 = 2$. Both matrices therefore are similar to a common diagonal matrix. So, they are similar.

b) (3 points) Fill in $\leq, =, \geq$. Let A be an arbitrary $n \times n$ matrix. The "number of eigenvalues" is the sum of all algebraic multiplicities of all eigenvalues.

The algebraic multiplicity of an eigenvalue of A is		its geometric multiplicity.
The number of complex eigenvalues of A is		n.
The number of real eigenvalues of A is		n.
The rank of A is		the number of non-zero real eigenvalues of A.

Solution:
$\geq$ because in general, geometric multiplicities are smaller or equal than algebraic multiplicities.
$=$ by the fundamental theorem of calculus.
$\leq$ because some eigenvalues could be complex
$\geq$ by the rank-nullity theorem. There could be complex eigenvalues however.

c) (2 points)

What is the algebraic multiplicity of the eigenvalue -1 for a 3×3 reflection at a line?	
What is the algebraic multiplicity of the eigenvalue -1 for a 3×3 reflection at a plane?	

d) (2 points)

What is the algebraic multiplicity of 1 for a 3×3 matrix which implements a projection onto a line?	
What is the algebraic multiplicity of 1 for a 3×3 matrix which implements a projection onto a plane?	

Solution:
c) 2 and 1.
d) 1 and 2.

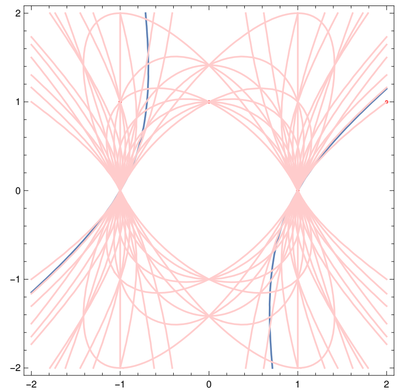
Problem 4) (10 points)

Find the equation of the form

$$x^2 + axy + by^2 = 1$$

that best fits the data points:

x	y
2	1
-1	1
1	0
0	1



Solution:

Writing down the equations and simplifying gives $A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. We have $A^TA = \begin{bmatrix} 5 & 1 \\ 1 & 3 \end{bmatrix}$ which has the inverse $(A^TA)^{-1} = \begin{bmatrix} \frac{3}{14} & -\frac{1}{14} \\ -\frac{1}{14} & \frac{5}{14} \end{bmatrix}$. Also,$A^Tb = [-6, -2]^T$. Now $(A^TA)^{-1}A^Tb = \begin{bmatrix} -\frac{8}{7} \\ -\frac{2}{7} \end{bmatrix}$. The best solution is $x^2 - (8/7)xy - (2/7)x^2$.

Problem 5) (10 points)

a) (2 points) Find the determinant of the "prime" matrix

$$A = \begin{bmatrix} 0 & 2 & 0 & 3 \\ 13 & 0 & 0 & 11 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix}.$$

Solution:
There is just one pattern with three upcrossing: $-13 \cdot 7 \cdot 5 \cdot 3 = 1365$.

b) (2 points) Find the determinant of the "count to 12" matrix

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 9 & 10 \\ 0 & 0 & 11 & 12 \end{bmatrix}.$$

Solution:
Partitioning the matrix reveals two 2×2 blocks. Thie determinant is the product of the determinants of the blocks. $(1 \cdot 6 - 2 \cdot 5)(9 \cdot 12 - 11 \cdot 10) = 8$.

c) (2 points) Find the determinant of the "11-1" matrix

$$C = \begin{bmatrix} 11 & 1 & 1 & 1 \\ 1 & 11 & 1 & 1 \\ 1 & 1 & 11 & 1 \\ 1 & 1 & 1 & 11 \end{bmatrix}.$$

Solution:
$C - 10$ has eigenvalues $0, 0, 0, 4$. So, C has eigenvalues $10, 10, 10, 14$. The determinant is $10^3 \cdot 14 = 14'000$.

d) (2 points) Find the determinant of the "Pascal triangle" matrix

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}.$$

Solution:
There is only one pattern. The number of upcrossings is 10 so, the determinant is 1.

e) (2 points) Find the determinant of the "mystery" matrix:

$$E = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 & 3 \\ 1 & 1 & 4 & 4 & 4 \\ 1 & 1 & 1 & 5 & 5 \\ 1 & 1 & 1 & 1 & 6 \end{bmatrix}.$$

Solution:

Row reduction. Subtract the first row from all others gives $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$. This matrix has determinant $5! = 120$.

Problem 6) (10 points)

The recursion $x(t+1) = 5x(t) - 2y(t), y(t+1) = 3x(t)$ leads to the discrete dynamical system

$$v(t+1) = Av(t) \ ,$$

where $A = \begin{bmatrix} 5 & -2 \\ 3 & 0 \end{bmatrix}$ and $v(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.

Find a closed form solution $v(t)$ with initial condition $v(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Solution:
The eigenvalues are $\lambda_1 = 3$ with eigenvector $v_1[1,1]^T$ and $\lambda_2 = 2$ with eigenvector $v_2 = [2,3]^T$. The initial vector $v(0) = [1,2]^T$ is now written as $v(0) = c_1 v_1 + c_2 v_2$ which gives $c_1 = -1, c_2 = 1$ so that the closed form solution is

$$v(t) = A^t v = (-1) \cdot 3^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T + 1 \cdot 2^t \begin{bmatrix} 2 \\ 3 \end{bmatrix}^T .$$

Problem 7) (10 points)

The matrix

$$H = \left[\begin{array}{cc|cccc|c} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{array}\right]$$

consists of three blocks: a 2×2 diagonal block $2I_2$, a 4×4 block and a 1×1 block $4I_1$.

a) (5 points) You are told that the eigenvalues $2, 2, 4$ in the first and third blocks also appear as eigenvalues in the middle block. Using this fact to find the remaining eigenvalue of the middle block.

Solution:
The trace of the middle block is 8. The last eigenvalue is 0. Actually, one can see its eigenvector $[1, 1, 1, 1]^T$.

b) (5 points) The matrix H can be written as $H = D^2$ where D is a "discrete Dirac operator"

$$D = \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 \end{bmatrix}$$

A "matter-anti-matter symmetry" assures you that for every non-zero eigenvalue λ of D there is also an eigenvalue $-\lambda$ of D. Use this fact to find all the 7 eigenvalues of the matrix D.

Solution:
We know the eigenvalues of H are $2, 2, 2, 2, 4, 4, 0$. If λ is an eigenvalue of D then λ^2 is an eigenvalue of $D^2 = H$. The eigenvalues are therefore the roots. Since we know that we have positive-negative pairs, the eigenvalues are $-\sqrt{2}, \sqrt{2}, -\sqrt{2}, \sqrt{2}, -2, 2, 0$.

Remark:
P.S. There is an interesting story here. The operator H is a "Laplacian" of a network. It is actually the Laplacian belonging to a new type of calculus, where not the individual components of space but individual pairs of space components are of interest. In this particular case, "space" consists of two points a, b only connected with a connection e. It is the simplest positive dimensional "world" you can build. If you are interested, continue reading.

Remark:
The interaction components in this world are given by the 7 possible pairs $(a, a), (a, e), (e, a), (b, e), (e, b), (b, b), (e, e)$ of intersecting components of zero dimensional parts a, b and one dimensional part e. This leads to the 7×7 matrix above). One can now do everything of the usual calculus and physics also on this new connection calculus. For example, in quantum mechanics, a Laplacian models the kinetic energy of a particle. It completely describes the motion of a single non-interacting particle. The equation $u_{tt} = -Hu$ is the wave equation which describes the motion of a particle in that world. It has the solution $\cos(Dt)u(0)+\sin(Dt)D^{-1}u'(0)$ (as D is not invertible, the inverse is taken on the eigenspaces of non-zero eigenvalues, this is called the "pseudo inverse". To check the identity, just differentiate twice. $d^2/dt^2 \cos(Dt) = -D^2 \cos(Dt) = -H \cos(Dt)$). If you define the complex vector $\psi(t) = u(t)+iD^{-1}u'(t)$, then because $e^{iDt} = \cos(Dt)+i\sin(Dt)$ (Euler), the solution of the wave equation can also be written as $e^{-iDt}\psi(0)$. In other words, the solution satisfies the Schrödinger equation $i\psi'(t) = D\psi(t)$. Now this works on any network. The point of the factorization $D^2 = H$ is that it allows to give explicit solutions of the wave equation on a network for example. It is quite easy to to write down the operator D for any "world". The factorization for the usual Laplacian $d^2/dx^2 + d^2/dy^2 + d^2/dz^2 = \mathrm{div}(\mathrm{grad})$ had first been done by Paul Dirac (who was motivated by physics too but also liked mathematical beauty and symmetry). It required Dirac matrices. In the discrete, we don't need that gymnastics. Things are much easier. The fact that physics on finite spaces is orders of magnitudes less technical than the physics in the continuum prompted physicists and philosophers to suggest space is actually discrete. Already Gottfried Leibniz started such thoughts. But it still remains just speculation. What counts in physics are affinities with real experiments: predictions of quantitative nature about physical processes which can be measured and verified. Until then, it is mathematics - or what is almost the same - poetry.

Remark:
We have given in part b) the information that if λ is an eigenvalue of D, then $-\lambda$ is an eigenvalue too. This is a consequence of what one calls **super symmetry** (in a purely mathematical sense, unlike in physics, mathematical super symmetry is always present. Physicists love symmetry so much that they believed for a long time that a stronger version holds fundamentally and that every particle has a super partner. Experiments at the Large Hadron Collider at CERN in the last decade have severely supressed this dream even so many still have hope.) In mathematics, there is no problem. It is a mathematical fact that it always appears and here is the proof: we can write $D = \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ with block matrices A. Now define the diagonal matrix $P = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ where I is the identity matrix. The simplest notion of mathematical super-symmetry (as coined and used by Ed Witten in purely mathematical frame works) are the matrix relations $L = D^2, P^2 = 1, DP + PD = 0$. You can check that these symmetries hold. Now, if $Dv = \lambda v$, then $PDPv = -DPPv = -Dv = -\lambda v$. We see that Pv is an other eigenvector, but with an eigenvalue $-\lambda$. The "particle" belonging to the energy $-\lambda$ is the "anti-particle" to the particle belonging to the energy λ. This zero-dimensional version of super symmetry is actually present in physics as we have seen anti-particles (Dirac got the Nobel prize in 1933 jointly with Erwin Schroedinger for that). Physicists have not seen super particles yet in particle accelerations. Maybe they don't exist.

Remark:
P.P.S. In the context of topology (one of the pillar subjects of math which are accessible after taking this course) one might be interested to see that this calculation actually computed the Betti vector $(b_0, b_1, b_2) = (0, 1, 0)$ in a **"cohomology"** defined by this one-dimensional network. This sounds fancy but it is just an impressive name for the kernels of some matrices. The number b_0 is the nullity of the first block, the number b_1 is the nullity of the second block, the number b_2 is the nullity of the third block. The number $b_0 - b_1 + b_2$ is the Wu characteristic of the network. It is defined as $\sum_{x \sim y} \omega(x)\omega(y)$, where $\omega(x) = (-1)^{dim(x)}$ and $x \sim y$ means that the two components are connected. In the case of the network under consideration, we have 7 summands and since $\omega(a) = \omega(b) = 1$ and $\omega(e) = -1$, we have a sum of 7 terms $\omega(a)\omega(a) + ... + \omega(e)\omega(e) = -1$. The fact that the alternating sum of the Betti numbers (an algebraically defined notion) is the same than the combinatorially computed notion is a higher order version of the Euler-Poincaré theorem (which does the same in traditional calculus). The fact that the Betti numbers can be expressed as the nullity of matrices is called Hodge theory. The proof in the discrete is not difficult at all and boils down essentially to the rank-nullity theorem. Things like that are the reason why the rank-nullity theorem is often called the "fundamental theorem of linear algebra".

Problem 8) (10 points)

a) (3 points) What are the eigenvalues of $H = 2I_8 + Q^2 + Q^{-2}$ given below?

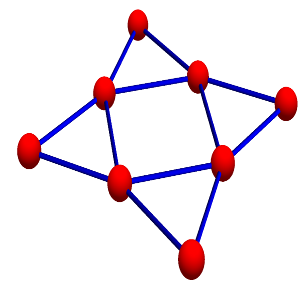
Solution:
$\lambda_k = e^{2\pi ik/8}$ for $k = 0, 1, 2, 3, 4, 5, 6, 7$.

b) (3 points) What are the eigenvectors of that same matrix H?

$$H = \begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 2 \end{bmatrix}, Q = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution:
$v_k = [1, \lambda_k, \lambda_k^2, \lambda_k^3, \ldots, \lambda_k^7]^T$. The eigenvectors v_k of H are the same than the eigenvectors of Q.

c) (2 points) By some magic related to the network seen to the right but not explained here, we can write $H = L - L^{-1}$, where the matrices are given below. You are told that $v = [0, -1, 0, 1, 0, -1, 0, 1]^T$ is an eigenvector of L with eigenvalue -1. Is this v also an eigenvector of L^{-1}? If yes, find the corresponding eigenvalue.



Solution:
If $Lv = \lambda v$, then $L^{-1}v = (1/\lambda)v$ and In the case $\lambda = -1$ this gives the eigenvalue -1 for L^{-1}.

d) (2 points) Is v also an eigenvector of $H = L - L^{-1}$? If yes, what is the corresponding eigenvalue?

$$L = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, L^{-1} = \begin{bmatrix} -1 & 1 & -1 & 0 & 0 & 0 & -1 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Solution:
Yes, it is. $(L - L^{-1})(v) = [\lambda - (1/\lambda)]v$. The corresponding eigenvalue is 0.

Remark:
By the way, an identity $H = L - L^{-1}$ can be written down for any one-dimensional network. The matrix H is in general more complicated in that the diagonal entries contain vertex degrees of the network. The matrix L is always a $0, 1$ matrix. It is easier mathematically do deal with it. The decomposition is useful for example to estimate the eigenvalues.

Problem 9) (10 points)

a) (2 points) Find the Q and R matrices in the QR-factorization of

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b) (2 points) Find the Q and R matrices in the QR-factorization of

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

c) (3 points) Find the Q and R matrices in the QR-factorization of AB.

d) (3 points) Find the Q and R matrices in the QR-factorization of

$$A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} .$$

Solution:
a) $A = QR = AI_6$. The R matrix is the identity matrix here.
b) $B = QR = I_6B$. The Q matrix is the identity matrix here.
c) $AB = QR$ gives $A = Q$ and $B = R$. The reason is that A is already orthogonal and B is triangular.
d) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$. As some have noticed, one can get B by switching just the two rows of A. The Q matrix, then undoes this.