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MWF 10 Hunter Spink
MWF 11 Matt Demers
MWF 11 Yu-Wen Hsu
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TTH 10 Ana Balibanu
TTH 10 Morgan Opie
TTH 10 Rosalie Belanger-Rioux
TTH 11:30 Philip Engel
TTH 11:30 Alison Miller

- Please fill in your name and mark your section.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) (20 points) True or False? No justifications are needed.

1) T F There is a real diagonalizable 3×3 matrix for which the algebraic multiplicity of an eigenvalue $\lambda = 2$ is larger than 1.

2) T F Two symmetric not-invertible 3×3 matrices A, B are similar if their trace agrees.

3) T F Every orthogonal 5×5 matrix has a real eigenvalue.

4) T F The real eigenvalues of a 4×4 matrix A do not change under row reduction.

5) T F Every real diagonalizable 3×3 matrix can be diagonalized using an orthogonal matrix S.

6) T F The eigenvalues of a 2×2 rotation matrix are always either 1 or -1.

7) T F The nullity of a $n \times n$ matrix A is the same as the nullity of A^T.

8) T F A discrete dynamical system $\vec{v}(t+1) = A\vec{v}(t)$ defined by a 2×2 matrix A is asymptotically stable if $A^8 = I_2/2$.

9) T F A 3×3 matrix for a reflection about a line never has trace 0.

10) T F The trace of a 5×5 orthogonal projection matrix is the dimension of the image.

11) T F If A and B are diagonalizable $n \times n$ matrices, then AB is diagonalizable.

12) T F The sum of the geometric multiplicities of a $n \times n$ matrix A is always at most n.

13) T F A 2×2 matrix B such that $\text{tr}(B) = 3$, $\text{tr}(B^2) = 5$ is diagonalizable.

14) T F There is a 3×3 matrix A such that $A + 2I$ has rank one and $A - 5I$ has rank one.

15) T F If A is a 3×3 matrix with eigenvalues $\lambda_1 = 3$, $\lambda_2 = 3$, $\lambda_3 = 7$, then $A - 3I$ has rank one and $A - 7I$ has rank two.

16) T F If A is a 10×2 matrix of rank 2, then the least square solution of $Ax = b$ is unique.

17) T F If A is an asymptotically stable 3×3 matrix, then A^T is asymptotically stable.

18) T F If a 3×3 matrix A is similar to C and a 3×3 matrix B is similar to D, then AB is similar to CD.

19) T F If a 3×3 matrix A is similar to the zero matrix 0 then A is equal to the zero matrix.

20) T F For any 3×3 matrices A, B we know that if A has the same determinant as B^3, then B has the same determinant as A^3.

Total

Problem 2) (10 points) No justifications are needed.

a) (2 points) Which matrices have the property that the system $x(t+1) = Ax(t)$ is asymptotically stable? We just write "stable" abbreviating asymptotically stable.

Matrix	stable	not stable
$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$		
$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$		
$\begin{bmatrix} 1/4 & 8 & 8 \\ 0 & 1/4 & 8 \\ 0 & 0 & 1/4 \end{bmatrix}$		

b) (2 points) Which identities hold for a non-invertible 3×3 matrix with eigenvalues α, β, γ and characteristic polynomial $f_A(\lambda)$?

Identity	always true	not always true
$\det(A) = 0$		
$\mathrm{tr}(A) = 0$		
$f_A(\alpha) = 0$		
$\dim(\ker(A)) = 0$		

c) (2 points) Which of the following complex numbers are real? Remember that we defined $w^z = e^{z \log(w)}$ and $\log(z) = \ln|z| + i \arg(z)$ with $0 \leq \arg(z) < 2\pi$ for any complex numbers $w \neq 0, z \neq 0$ and where $\arg(z)$ is the angle so that $z = |z|e^{i \arg(z)}$.

Number	is real	is not real
$\log(i\pi)$		
$e^{i\pi}$		
$\log(e^{i\pi})$		
$e^{i\pi/2}$		

d) (2 points) Which type of matrices are always diagonalizable **over the real numbers**?

Type of matrix	always diagonalizable over the reals	not necessarily diagonalizable
symmetric		
orthogonal		
horizontal shear		
dilation		

e) (2 points) If S is a 4×4 matrix whose columns are given by an eigenbasis of a matrix A which has eigenvalues $0, 1, 2, 3$, then

The statement	is always true	can be false
A is invertible		
S is invertible		
$A + I_4$ is invertible		
$A - I_4$ is invertible		

Problem 3) (10 points) No justifications are needed

a) (2 points) A, B are arbitrary 3×3 matrices. Each of the two has distinct eigenvalues meaning that the algebraic multiplicity of each eigenvalue is 1. We write $A \sim B$ to indicate that A is similar to B.

The statement	implies $A = B$	implies $A \sim B$
A, B have the same eigenvalues		
A, B have same eigenvectors		
A, B have same eigenvalue,eigenvector pairs		

b) (2 points) Assume A is an invertible and diagonalizable 2×2 matrix. Which matrices are diagonalizable too?

A^2	
A^T	
A^{-1}	
$2A + I_2$	

c) (2 points) Fill in each of the two cases the Q and R matrices giving the QR factorization $A = QR$ of A:

A	Q	R
$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$		
$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$		

d) (2 points) Which of the following matrices A has the property that

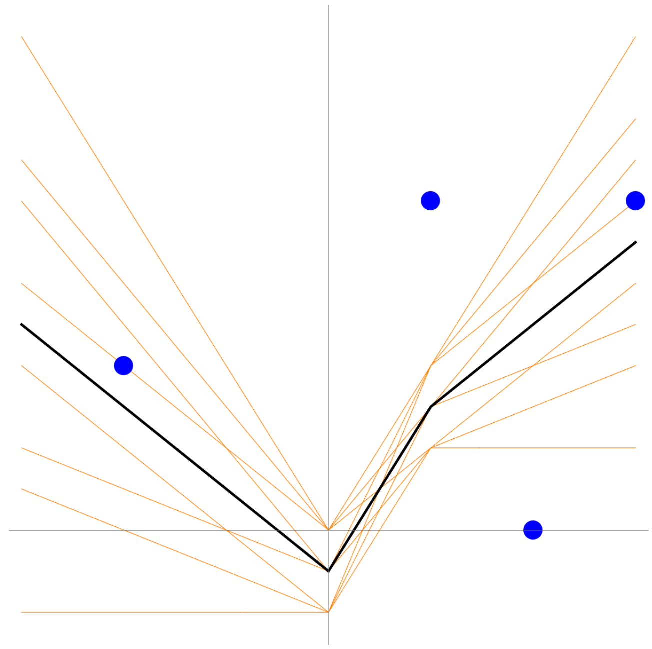
$$A[x(t), x(t-1), x(t-2)]^T = [x(t) + x(t-1) + x(t-2), x(t), x(t-1)]^T .$$

$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	
$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	
$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	

e) (2 points) Exactly one of the three statements is **not** part of the spectral theorem. Which one?

A symmetric matrix has an orthonormal eigenbasis	
A real symmetric matrix has real eigenvalues	
A matrix with real eigenvalues has an orthonormal eigenbasis	

Problem 4) (10 points)



Find the function

$$a|x| - b|x-1| = y$$

which is the best fit for the data

x	y
1	2
3	2
-2	1
2	0

Problem 5) (10 points)

a) (2 points) Find the determinants of

$$A(2) = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}, \quad A(3) = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 5 & 2 \\ 0 & 2 & 5 \end{bmatrix}.$$

b) (2 points) Use Laplace expansion to get the determinant of

$$A(4) = \begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & 5 & 2 & 0 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix}.$$

c) (2 points) Find the determinant of

$$B = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 & 0 \\ 0 & 0 & 2 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 5 & 0 \\ 0 & 0 & 0 & 0 & 2 & 5 \end{bmatrix}.$$

d) (2 points) Find the determinant of

$$C = \begin{bmatrix} 5 & 2 & 2 & 2 & 2 \\ 2 & 5 & 2 & 2 & 2 \\ 2 & 2 & 5 & 2 & 2 \\ 2 & 2 & 2 & 5 & 2 \\ 2 & 2 & 2 & 2 & 5 \end{bmatrix} .$$

e) (2 points) Find the determinant of

$$D = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 & 0 \end{bmatrix} .$$

Problem 6) (10 points)

The recursion $x(t+1) = 5x(t) - 4x(t-1)$ can be written as $v(t+1) = Av(t)$.

a) (3 points) Fill in the 2×2 matrix A to make this a recursion

$$\begin{bmatrix} x(t+1) \\ x(t) \end{bmatrix} = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-1) \end{bmatrix} .$$

b) (3 points) Find the eigenvalues and eigenvectors of A.

c) (4 points) Write the initial condition $\vec{v}(0) = \begin{bmatrix} x(1) \\ x(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ as a combination of eigenvectors and use this to write down the **closed form solution** for $\vec{v}(t)$.

P.S. The recursion appearing here could have been useful to compute the determinants $A(n)$ in problem 5a) and 5b).

Problem 7) (10 points)

The matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

has the eigenbasis

$$\left\{ \begin{bmatrix} \sqrt{2} \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -\sqrt{2} \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\} .$$

a) (3 points) Find the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of A.

b) (3 points) Find the eigenvalues μ_1, μ_2, μ_3 of the inverse

$$A^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} .$$

c) (2 points) What are the eigenvectors of A^{-1}?

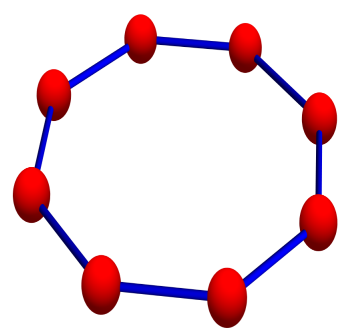
d) (2 points) Check the boxes which lead to true identities "left expression = expression above":

	tr(A)	det(A)
$\lambda_1 + \lambda_2 + \lambda_3$		
$\lambda_1 \lambda_2 \lambda_3$		

Problem 8) (10 points)

The Laplacian of the circular graph C_8 is

$$B = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} .$$



The matrix B can be written as $2I - A - A^{-1}$, where A is an orthogonal matrix.

a) (5 points) What are the eigenvalues of B?

b) (5 points) What are the eigenvectors of B?

Problem 9) (10 points)

a) (4 points) Find the QR factorization of

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b) (4 points) Find the QR factorization of

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

c) (2 points) Assume A is an orthogonal 3×3 matrix. What is

$$P = A(A^T A)^{-1} A^T \text{ ?}$$

Problem 10) (10 points)

Assume A is an invertible 3×3 matrix.

a) (2 points) Is A^T necessarily invertible?
b) (2 points) Is A^{-1} necessarily invertible?
c) (2 points) Is $A + A^T$ necessarily invertible?
d) (2 points) Is $A + I_3$ necessarily invertible?
e) (2 points) If $A = QR$ is the QR decomposition. Are both R and Q invertible?