Name:

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MWF 10 Hunter Spink
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MWF 11 Yu-Wen Hsu
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TTH 11:30 Philip Engel
TTH 11:30 Alison Miller

- Please fill in your name and mark your section.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- All matrices are real matrices unless specified otherwise.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) True or False? No justifications are needed.

1)	T	F	The matrix $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ has the eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
2)	T	F	If a 2×2 matrix A has trace 2018, then the trace of $B = A - 2018 I_2$ is zero.
3)	T	F	If two 2×2 matrices A and B each have an eigenvalue of 2018, then the matrix $A - B$ has an eigenvalue of 0.
4)	T	F	If A is invertible, then $A^T A$ is invertible.
5)	T	F	There is an invertible matrix A such that A^T is not invertible.
6)	T	F	Every nonzero 2×2-matrix A satisfying $A^2 = 0$ can be diagonalized.
7)	T	F	Let A be a 1×2-matrix. If the trace $\mathrm{tr}(A^T A) = 0$ then $A = 0$.
8)	T	F	Every orthogonal projection has an eigenbasis.
9)	T	F	If A and B are similar, then they have the same rank.
10)	T	F	Let A, B be two 2×2 rotation matrices with the same trace. Then they have the same eigenvalues.
11)	T	F	The trace of a matrix A does not change under row reduction.
12)	T	F	A discrete dynamical system $\vec{v}(t+1) = A\vec{v}(t)$ with 2×2 matrix A is stable if $\det(A) < 1$.
13)	T	F	Let A be a 2×2 reflection dilation matrix, reflecting at a line. Then the trace of A is zero.
14)	T	F	If A is an arbitrary $n \times n$ matrix, then $A + A^T$ is diagonalizable.
15)	T	F	The sum of the algebraic multiplicities of a symmetric $n \times n$ matrix A can be smaller than n.
16)	T	F	A 2×2 matrix A such that $\mathrm{tr}(A) = 3$, $\det(A) = 5$ is diagonalizable over the reals.
17)	T	F	There is a 4×4 matrix A such that $A + I$ has rank two and $A - 2I$ has rank one.
18)	T	F	If A is a 2×2 matrix with eigenvalues $\lambda_1 = 5$, $\lambda_2 = 5$, then $A - 5I_2$ has rank two.
19)	T	F	The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ is diagonalizable.
20)	T	F	If A, B are diagonalizable and have the same eigenvalues, then $A - B$ has all eigenvalues zero.

	Total

Problem 2) (10 points) No justifications are needed.

a) (3 points) Which matrices A have the property that the discrete dynamical system

$$v(t+1) = Av(t)$$

is asymptotically stable?

Matrix A	asymptotically stable	not asymptotically stable
$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$		
$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$		
$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$		

b) (2 points) Which matrix A encodes the discrete dynamical system $v(t+1) = Av(t)$ if

$$v(t+1) = [x(t+1), x(t)]^T = [2x(t) + x(t-1), x(t)]^T = Av(t)\ ?$$

Check exactly one matrix

Matrix	
$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$	

Matrix	
$A = \begin{bmatrix} 2 & 1 \end{bmatrix}$	

Matrix	
$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$	

c) (2 points) Which of the following complex numbers are real? Remember that we defined $w^z = e^{z\log(w)}$ and $\log(z) = \ln|z| + i\,\theta$ with $0 \leq \theta < 2\pi$ for any complex numbers $w \neq 0$, $z \neq 0$, where θ is the angle so that $z = |z|e^{i\,\theta}$.

Number	is real	is not real
$i\log(i)$		
$i+i$		

Number	is real	is not real
i^2		
i^i		

d) (3 points) For each type of matrix check every box such that the corresponding property always holds for that type of matrix. By diagonalizable, we mean "diagonalizable with all eigenvalues being real".

	invertible	diagonalizable	symmetric	real eigenvalues
Projection matrix				
Shear matrix				
Rotation matrix				
Reflection matrix				
$A^2 = 0, A \neq 0$				
Diagonal matrix				

Problem 3) (10 points) No justifications are needed

a) (3 points) The following 6 matrices can be grouped into 3 pairs of similar transformations. Find these three pairs.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}, E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

b) (3 points) Fill in $\leq, =, \geq$. Let A be an arbitrary $n \times n$ matrix. The "number of eigenvalues" is the sum of all algebraic multiplicities of all eigenvalues.

The algebraic multiplicity of an eigenvalue of A is		its geometric multiplicity.
The number of complex eigenvalues of A is		n.
The number of real eigenvalues of A is		n.
The rank of A is		the number of non-zero real eigenvalues of A.

c) (2 points)

What is the algebraic multiplicity of the eigenvalue -1 for a 3×3 reflection at a line?	
What is the algebraic multiplicity of the eigenvalue -1 for a 3×3 reflection at a plane?	

d) (2 points)

What is the algebraic multiplicity of 1 for a 3×3 matrix which implements a projection onto a line?	
What is the algebraic multiplicity of 1 for a 3×3 matrix which implements a projection onto a plane?	

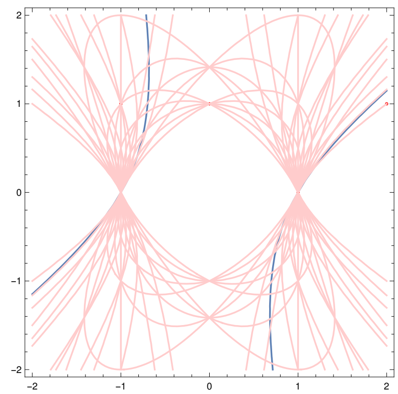
Problem 4) (10 points)

Find the equation of the form

$$x^2 + axy + by^2 = 1$$

that best fits the data points:

x	y
2	1
-1	1
1	0
0	1



Problem 5) (10 points)

a) (2 points) Find the determinant of the "prime" matrix

$$A = \begin{bmatrix} 0 & 2 & 0 & 3 \\ 13 & 0 & 0 & 11 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix} .$$

b) (2 points) Find the determinant of the "count to 12" matrix

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 9 & 10 \\ 0 & 0 & 11 & 12 \end{bmatrix} .$$

c) (2 points) Find the determinant of the "11-1" matrix

$$C = \begin{bmatrix} 11 & 1 & 1 & 1 \\ 1 & 11 & 1 & 1 \\ 1 & 1 & 11 & 1 \\ 1 & 1 & 1 & 11 \end{bmatrix} .$$

d) (2 points) Find the determinant of the "Pascal triangle" matrix

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} .$$

e) (2 points) Find the determinant of the "mystery" matrix:

$$E = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 & 3 \\ 1 & 1 & 4 & 4 & 4 \\ 1 & 1 & 1 & 5 & 5 \\ 1 & 1 & 1 & 1 & 6 \end{bmatrix} .$$

Problem 6) (10 points)

The recursion $x(t+1) = 5x(t) - 2y(t), y(t+1) = 3x(t)$ leads to the discrete dynamical system

$$v(t+1) = Av(t) ,$$

where $A = \begin{bmatrix} 5 & -2 \\ 3 & 0 \end{bmatrix}$ and $v(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.

Find a closed form solution $v(t)$ with initial condition $v(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Problem 7) (10 points)

The matrix

$$H = \left[\begin{array}{cc|cccc|c} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{array}\right]$$

consists of three blocks: a 2×2 diagonal block $2I_2$, a 4×4 block and a 1×1 block $4I_1$.

a) (5 points) You are told that the eigenvalues 2, 2, 4 in the first and third blocks also appear as eigenvalues in the middle block. Using this fact to find the remaining eigenvalue of the middle block.

b) (5 points) The matrix H can be written as $H = D^2$ where D is a "discrete Dirac operator"

$$D = \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 \end{bmatrix}$$

A "matter-anti-matter symmetry" assures you that for every non-zero eigenvalue λ of D there is also an eigenvalue $-\lambda$ of D. Use this fact to find all the 7 eigenvalues of the matrix D.

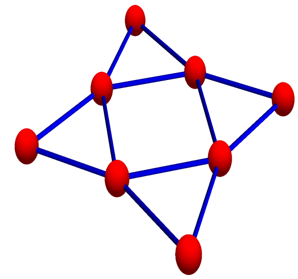
Problem 8) (10 points)

a) (3 points) What are the eigenvalues of $H = 2I_8 + Q^2 + Q^{-2}$ given below?

b) (3 points) What are the eigenvectors of that same matrix H?

$$H = \begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 2 \end{bmatrix}, Q = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

c) (2 points) By some magic related to the network seen to the right but not explained here, we can write $H = L - L^{-1}$, where the matrices are given below. You are told that $v = [0, -1, 0, 1, 0, -1, 0, 1]^T$ is an eigenvector of L with eigenvalue -1. Is this v also an eigenvector of L^{-1}? If yes, find the corresponding eigenvalue.



d) (2 points) Is v also an eigenvector of $H = L - L^{-1}$? If yes, what is the corresponding eigenvalue?

$$L = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, L^{-1} = \begin{bmatrix} -1 & 1 & -1 & 0 & 0 & 0 & -1 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Problem 9) (10 points)

a) (2 points) Find the Q and R matrices in the QR-factorization of

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b) (2 points) Find the Q and R matrices in the QR-factorization of

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

c) (3 points) Find the Q and R matrices in the QR-factorization of AB.

d) (3 points) Find the Q and R matrices in the QR-factorization of

$$A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}.$$