Homework 31: Partial differential equations

This is the last homework. It is due on the last day of classes: Wednesday April 25, respectively on Tuesday, April 24, 2018.

1 Solve the heat equation $f_t = 2018 f_{xx}$ on $[0, \pi]$ with the initial condition $f(x, 0) = 0$ if $x \in [0, \pi/2]$ and $f(x) = \sin(2x)$ if $x \in [\pi/2, \pi]$.

Solution:

In this course, we always compute the sin series when dealing with PDEs. When computing the series, we only have to integrate from $\pi/2$ to π. Use the formula $2\sin(2x)\sin(nx) = (\cos((n+2)x) + \cos(n-2)x))$ to get the coefficients are

$$b_n = (2/\pi)\int_{\pi/2}^{\pi} \sin(2x)\sin(nx)\,dx$$

$$= (1/\pi)\int_{\pi/2}^{\pi} \cos((n+2)x) + \cos(n-2)x)\,dx$$

$$= -\sin((n+2)\pi/2)/(\pi(n+2)) - \sin((n-2)\pi/2;)/(\pi(n-2)) \ .$$

This could be simplified to $4\sin(n\pi/2)/((n^2-4)\pi)$ but the above expression is fine. Now, having the Fourier series, we just have to plug in the $e^{-\lambda_n t}$ parts. The solution is $\Sigma_{n=1}^{\infty} b_n e^{-2018 n^2 t}\sin(nx)$.

2 Solve the partial differential equation $u_t = 3u_{xxxxxx} + 5u_{xx}$ with initial condition $u(0, x) = 21x$.

Solution:
The initial condition is $b_n = (2/\pi) \int_0^\pi 21x \sin(nx)\, dx$ which can be computed using integration by parts. The answer is $b_n = 42(-1)^{n+1/n}$. Since we have on the right hand side the operator $D^6 + D^2$ which has the eigenvalues $-3n^6 - 5n^2$, we have the solution $\sum_n b_n e^{(-3n^6-5n^2)t} \sin(nx)$.

3 A piano string is fixed at the ends $x = 0$ and $x = \pi$ and is initially undisturbed $u(x, 0) = 0$. The piano hammer induces an initial velocity $u_t(x, 0) = g(x)$ onto the string, where $g(x) = \sin(3x)$ on the interval $[0, \pi/2]$ and $g(x) = 0$ on $[\pi/2, \pi]$. How does the string amplitude $u(x, t)$ move, if it follows the wave equation $u_{tt} = u_{xx}$?

Solution:
The Fourier series has the coefficients $b_n = (2/\pi) \int_0^{\pi/2} \sin(3x)\sin(nx) \;\; dx = (2n\cos(n\pi/2)/(\pi(n^2 - 9))$. The solution is

$$u(x, t) = \sum_n b_n \sin(nt)/n \sin(nx) \; .$$

4 A laundry line is excited by the wind. It satisfies the differential equation $u_{tt} = u_{xx} + \cos(t) + \cos(3t)$. Assume that the amplitude u satisfies initial position $u(x, 0) = x$ and $u_t(x, 0) = 4\sin(5x) + 10\sin(6x)$. Find the function $u(x, t)$ which satisfies the differential equation.
Hint. First find the general homogeneous solution $u_{homogeneous}$ of $u_{tt} = u_{xx}$ for an odd u then a particular solution $u_{particular}(t)$ which only depends on t. Finally fix the Fourier coefficients.

Solution:
The homogeneous solution can be written down immediately since the Fourier series is already given for the velocity. We also know the Fourier series for the position already because we have done it in the handout and in a previous homework again! We get

$$u_{homogeneous} = \sum_n ((-1)^{n+1}/n)\cos(nt)\sin(nx) + 4\sin(5t) sin(5x)/5 + 10\,\mathrm{si}$$

To get a particular solution, we have to solve $u_{tt} = \cos(t) + \cos(3t)$ which has the solution $-\cos(t) - \cos(3t)/9 + C_1 t + C$. Because the initial position satisfies $u(0,0) = 0$ and $u'(x,0) = 0$ by assumption, we have $C_1 = 0$ and $C = 1 + 1/9$. The general solution is

$$\sum_n ((-1)^{n+1}/n)\cos(nt)\sin(nx) + 4\sin(5t) sin(5x)/5 + 10\sin(6t)\sin(6x)/6$$

5 We have looked at 4 different types of differential equations. Systems of linear differential equations $x' = Ax$, nonlinear equations $x' = f(x,y), y' = g(x,y)$, inhomogeneous equations $p(D)f = g$ as well as partial differential equations like $u_t = D^2 u$ and the wave equation $u_{tt} = D^2 u$. Give an original example of each type (it should not have appeared in any homework nor handout).

Solution:
Example: a) $x' = 3x + y, y' = 2x + 2y$,
b) $x' = x^2 + x + yx, y' = x - y^4$,
c) $f''' + f'' + f' + f = \sin(x)$,
d) $u_{tt} = D^2 - D^6 + \sin(t) = 0$.

Partial differential equations

Solving a PDE means to find a function f on the interval $[0, \pi]$. We write it as a sin-series which means that we only need to compute the b_n using the formula

$$\frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) \, dx \; .$$

This is justified as we can think of f continued as $f(-x) = -f(x)$ on $[-\pi, 0]$ The temperature distribution $f(x,t)$ in a metal bar $[0, \pi]$ satisfies the **heat equation**

$$f_t(x,t) = \mu f_{xx}(x,t) = D^2 f(x,t)$$

Here μ is a positive constant which depends on the material. The height of a string $f(x,t)$ at time t and position x on $[0, \pi]$ satisfies the **wave equation**

$$f_{tt}(x,t) = c^2 f_{xx}(x,t) = c^2 D^2(f)(x,t)$$

Here c is a positive constant which tell how fast the waves move. All problems are solved by diagonalizing D^2 using a Fourier basis. For heat, write the initial condition as a Fourier series and write down the solution. For wave, write both the initial condition $f(0,x)$ as well as the initial velocity $f_t(0,x)$ as a Fourier series and write down the solutions.