Homework 26: Nonlinear systems

This homework is due on Wednesday, April 11, respectively on Thursday, April 12, 2018. There is a handout for this material on the website.

1 Analyze the system

$$\frac{dx}{dt} = 2x - x^2 + xy$$
$$\frac{dy}{dt} = 4y - xy - y^2$$

It is an interaction model of species so that we only look at $x \geq 0, y \geq 0$.

Solution:
The equations factor as $\frac{dx}{dt} = x(2 - x + y), \frac{dy}{dt} = y(4 - x - y)$. Draw the nullclines which are the x-axes, y-axes, and two lines intersecting at $(3, 1)$. The stationary points are $(0, 0), (3, 1), (2, 0)$ and $(0, 4)$. The Jacobian matrix in general

$$\begin{bmatrix} -2x + y + 2 & x \\ -y & -x - 2y + 4 \end{bmatrix}$$

Solution:
We compute it at each critical point and see the eigenvalues:

$$(0,4): \begin{bmatrix} 6 & 0 \\ -4 & -4 \end{bmatrix} \quad (6,-4)$$

$$(2,0): \begin{bmatrix} -2 & 2 \\ 0 & 2 \end{bmatrix} \quad (-2,2)$$

$$(3,1): \begin{bmatrix} -3 & 3 \\ -1 & -1 \end{bmatrix} \quad -2 \pm i\sqrt{2}.$$

$$(0,0): \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \quad (2,4)$$

We see that the system spirals in at $(3,1)$ (stable point), has two hyperbolic unstable points (with a stable and unstable direction) and an unstable point (two unstable directions). The analysis of the system is now clear; in the quadrant $x > 0, y > 0$, everything converges to $(3,1)$. On the positive x-axis, everything goes to $(2,0)$; on the positive y-axis, everything goes to $(0,4)$.

2 We analyze the system

$$\begin{aligned} \frac{dx}{dt} &= x(1 - x + ky - k) \\ \frac{dy}{dt} &= y(1 - y + kx - k) \end{aligned}$$

in the cases $k = 2$ and $k = 0$ as well as $k = -2$. Again, as this is a population model, we only look at $x \geq 0, y \geq 0$.

Solution:

The null-clines are $x = 0, y = 0, (1 - x + ky - k) = 0$ and $(1 - y + kx - k) = 0$. Intersecting different null-clines gives the equilibrium points $(0, 0), (1, 1), (0, 1 - k), (1 - k, 0)$. The Jacobian matrix is

$$\begin{bmatrix} 1 - 2x + ky - k & kx \\ ky & 1 - 2y + kx - k \end{bmatrix}$$

Solution:

We compute it at each of the critical points and see the eigenvalues:

$(0, 1 - k)$ $$\begin{bmatrix} 1 - k^2 & 0 \\ (1 - k)k & k - 1 \end{bmatrix}$$

eigenvalues: $-1 + k, 1 - k^2$

$(1, 1)$ $$\begin{bmatrix} -1 & k \\ k & -1 \end{bmatrix}$$

eigenvalues: $-1 - k, -1 + k$

$(0, 0)$ $$\begin{bmatrix} 1 - k & 0 \\ 0 & 1 - k \end{bmatrix}$$

eigenvalues: $1 - k, 1 - k$

$(1 - k, 0)$ $$\begin{pmatrix} k - 1 & (1 - k)k \\ 0 & 1 - k^2 \end{pmatrix}$$

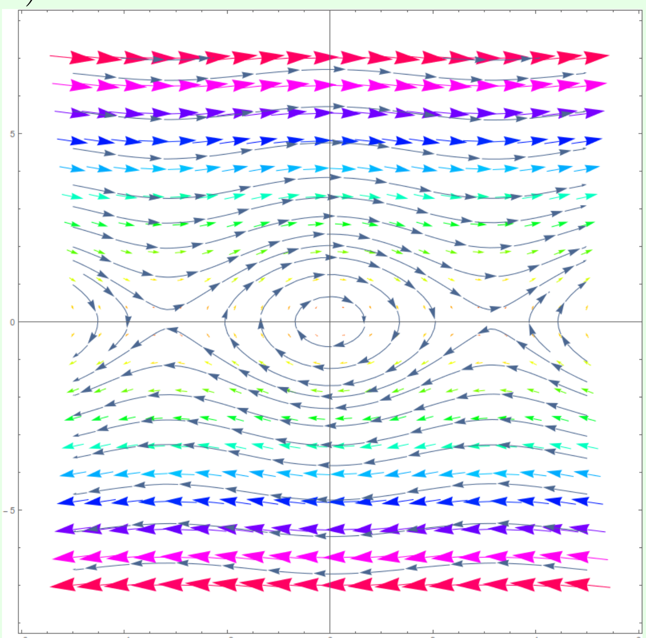
eigenvalues: $-1 + k, 1 - k^2$

The point $(0, 0)$ is stable when $k > 1$ like $k = 2$. The point $(1, 1)$ is stable when $-1 < k < 1$ like $k = 0$ and the points $(0, 1 - k)$ and $(1 - k, 0)$ are only stable when $k < -1$ like $k = -2$.

3 Analyze the frictionless pendulum

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -2\sin(x)\ ,\end{aligned}$$

Solution:
a)



The trajectories are shown above. The equilibrium points are $(k\pi, 0)$, where k is an integer. The Jacobian matrix is $J = \begin{bmatrix} 0 & 1 \\ -2\cos(k\pi) & 0 \end{bmatrix}$, which has eigenvalues of $\pm i\sqrt{c}$ for even k and $\pm\sqrt{c}$ for odd k. The points $((2k+1)\pi, 0)$ are hyperbolic points; they are not stable. The points $(2k, \pi, 0)$ are also not stable, trajectories nearby rotate about these points but do not converge to them.

4 Analyze the system

$$\frac{dx}{dt} = x^2 + y^2 - 1$$

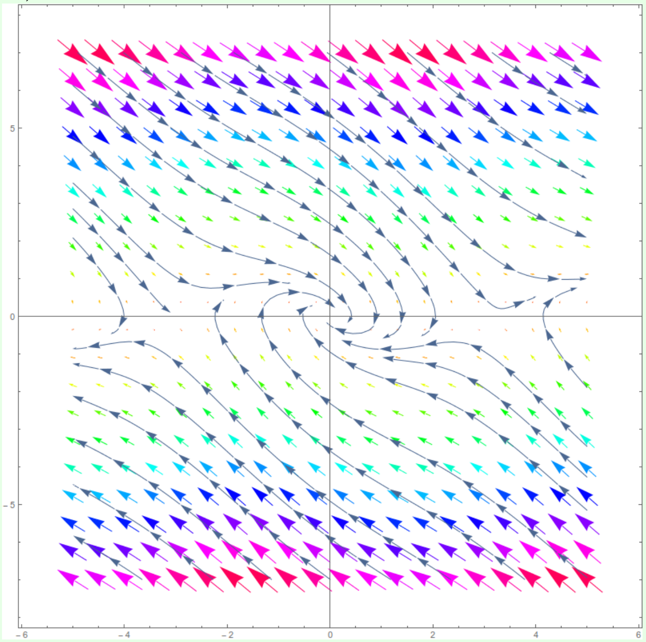
$$\frac{dy}{dt} = xy$$

Solution:
The nullclines are the unit circle and the union of the x and y axes. Taking the intersection gives the equilibrium points $(-1,0),(1,0),(0,1),(0,-1)$. We have the Jacobian matrix $J = \begin{bmatrix} 2x & 2y \\ y & x \end{bmatrix}$, with $\mathrm{Tr}(J) = 3x$ and $\det(J) = 2x^2 - 2y^2$. We have $\det(J) > 0$ at $(1,0)$ and $(-1,0)$ with $\mathrm{Tr}(J) > 0$ at $(1,0)$ and $\mathrm{Tr}(J) < 0$ at $(-1,0)$. That is to say $(-1,0)$ is a stable equilibrium point and the other three are not stable.

5 Analyze the pendulum with friction

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -\sin(x) - y \ .$$

Solution:
a)



The Jacobian matrix is given by $J = \begin{bmatrix} 0 & 1 \\ -\cos(x) & -1 \end{bmatrix}$.

Solution:

The equilibrium points are the same as in problem 3; the points $(k\pi, 0)$, where k is an integer. The points $(2k\pi, 0)$ are now stable with eigenvalues $(-1 \pm i\sqrt{3})/2$. The trajectories spiral in at those points, as we have a complex eigenvalue with negative real part. The trajectories at $((2k+1)\pi, 0)$ are all hyperbolic as the Jacobean matrix now has the eigenvalues $(-1 \pm \sqrt{5})/2$.

Nonlinear systems

Differential equations $x' = f(x,y), y' = g(x,y)$ generalize the linear case $x' = ax + by, y' = cx + dy$. To analyze such systems when f, g are not linear, we draw **phase portraits**. The curves where $f(x,y) = 0$ or $g(x,y) = 0$ are called nullclines. They intersect in **equilibrium points**. These are points where $x' = 0, y' = 0$. We can use linear algebra to analyze the system near such an equilibrium point (a,b). The matrix $A = \begin{bmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{bmatrix}$ is called the **Jacobian matrix**. The linear system $v' = Av$ is called the **linearization** at (x_0, y_0). If this linear system is stable, the equilibrium point is stable. In terms of the original nonlinear system, an equilibrium point (x_0, y_0) is stable if all trajectories starting sufficiently close to (x_0, y_0) tend to it as $t \to \infty$.

Making an **analysis** of the system consists of 1) finding the nullclines and equilibria 2) determine the stability of the equilibria 3) drawing the phase portrait of the system 4) analyzing the possible behaviors of the trajectories.