Homework 22: Stability

This homework is due on Friday, March 30, respectively on Tuesday, April 3, 2018.

1 Determine the stability of the dynamical system $x(t+1) = Ax(t)$:

a) $\begin{bmatrix} 0.2 & 0.3 & 0.4 \\ 0.2 & 0.3 & 0.4 \\ 0.2 & 0.3 & 0.4 \end{bmatrix}$.

b) $\begin{bmatrix} 0.9 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$.

Solution:

a) There is a two dimensional kernel and the trace is 0.9, so it is clear that the eigenvalues are $0, 0, 0.9$. We conclude that the system is stable.

b) A has an eigenvalue 0.9 and two complex conjugate eigenvalues $1 + 2i, 1 - 2i$. The system is not stable since the absolute value of the eigenvalues is not smaller than 1.

2 For which constants a is the system $x(t + 1) = Ax(t)$ stable?

a) $A = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$.

b) $A = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}$. c) $A = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$.

Solution:

a) The trace is zero, the determinant is $-a^2$. We have stability if $|a| < 1$. You can also see this from the eigenvalues, $a, -a$.

b) Look at the trace-determinant plane. The trace is a, the determinant -1. This is nowhere inside the stability triangle so that the system is always unstable.

c) The eigenvalues are $0, 2a$. The system is stable if and only if $|2a| < 1$ which means $|a| < 1/2$.

3 For which real values k does the drawing rule

$$\begin{aligned} x(t+1) &= x(t) - ky(t) \\ y(t+1) &= y(t) + kx(t+1) \end{aligned}$$

produce trajectories which are ellipses? Write the system first as a discrete dynamical system $v(t+1) = Av(t)$ and look for the k for which the eigenvalues λ_k satisfy $|\lambda_k| = 1$.

Solution:

First plug in the first equation into the second to get a recursion which only involves $x(t), y(t)$ on the right hand side. The recursion is $v(t+1) = Av(t)$ with $A = \begin{bmatrix} 1 & -k \\ k & 1-k^2 \end{bmatrix}$. The eigenvalues are the solutions of the equation $\lambda^2 + (k^2 - 2)\lambda + 1$ which is $(2 - k^2) + k\sqrt{k^2 - 4})/2$. The determinant is 1. We look for k's for which the eigenvalues are on the unit circle This happens if there is an imaginary part. This is the case for $-2 < k < 2$. Actually, for $k = 0, k = 2, k = -2$ the eigenvalue are real. These are limiting case. As an engineer, one would have to make sure that the rotation angle is irrational as otherwise, the ellipse is not drawn completely.

4 Find the eigenvalues of

$$A = \begin{bmatrix} 0 & a & b & c & 0 & 0 \\ 0 & 0 & a & b & c & 0 \\ 0 & 0 & 0 & a & b & c \\ c & 0 & 0 & 0 & a & b \\ b & c & 0 & 0 & 0 & a \\ a & b & c & 0 & 0 & 0 \end{bmatrix}$$

Where a, b and c are arbitrary constants. Verify that the discrete dynamical system is stable for $|a| + |b| + |c| < 1$.

Solution:
Write $A = aQ + bQ^2 + cQ^3$, where Q is the shift matrix with characteristic polynomial $\lambda^6 - 1$. The eigenvalues are $ae^{2\pi ik/6} + be^{2\pi i2k/6} + ce^{2\pi i3k/6}$ for $k = 0, .., 5$. The system is stable if all the eigenvalues are smaller than 1 in absolute value. This is the case if $|a| + |b| + |c| < 1$.

5 In the following, answer each question with a short explanation. We say A is stable if the origin $\vec{0}$ is a stable equilibrium.

a) True or false: the identity matrix is stable.
b) True or false: the zero matrix is stable.
c) True or false: every horizontal shear is stable.
d) True or false: any reflection matrix is stable.
e) True or false: A is stable if and only if A^T is stable.
f) True or false: A is stable if and only if A^{-1} is stable.
g) True or false: A is stable if and only if $A + 1$ is stable.
h) True or false: A is stable if and only if A^2 is stable.
i) True or false: A is stable if $A^2 = 0$.
j) True or false: A is unstable if $A^2 = A$.
k) True or false: A is stable if A is diagonalizable.

Solution:

a) False: The identity is not

b) True: The zero matrix has 0 eigenvalues. It is stable.

c) False: The shear has an eigenvalue 1.

d) False: A reflection preserves length. It is not possible that $A^n x \to 0$.

e) True: the matrices A and A^T have the same eigenvalues.

f) False: the matrix A^{-1} has the eigenvalue $1/\lambda$ if A has the eigenvalue λ. For $\lambda = 1/2$ the inverse has an eigenvalue 2.

g) False: Take A=0 which is stable but A=1 is not stable.

h) True: $|\lambda| < 1$ if and only if $|\lambda^2| < 1$.

i) True: If $A^2 = 0$, then A^2 has zero eigenvalues. Therefore also A has zero eigenvalues. We see that A is stable.

j) False: $A = 0$ satisfies the equation $A^2 = A$ but we have stability.

k) False: take a diagonal matrix which has diagonal entries 2. This is unstable but diagonalizable.

Stability

A discrete dynamical system $x(t+1) = Ax(t)$ is **asymptotically stable** if $x(t) \to 0$ as $t \to \infty$ for all initial conditions $x(0)$. (If we say "stable" we always mean asymptotically stable). The main result covered in this section is that a system is asymptotically stable if and only all eigenvalues of A have absolute value $|\lambda_j| < 1$. For example, a rotation dilation A with first column $Ae_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ is stable if and only if $a^2 + b^2 < 1$. We often just say "A is stable" rather than "the origin is stable for the discrete dynamical system $x \mapsto Ax$".