Homework 21: Complex Eigenvalues

This homework is due on Wednesday, March 28, respectively on Thursday, March 29, 2018.

1 a) For $z = 1 + 4i$. Find $z + z^2 + z^3$.
b) The log of a nonzero complex number $re^{i\theta} = r\cos(\theta) + ir\sin(\theta)$, where $r > 0$ and $0 \leq \theta < 2\pi$, is defined as $\log(re^{i\theta}) = \log r + i\theta$. Find the logarithm of $e \cdot i$. (”Ei” is the product of e and i. It means ”Egg” in German.)
c) Using logarithms we can define $w^z = e^{z \log w}$.
What is i^i, the “eye for an eye” number?

Solution:
a) $-61 - 40i$
b)$1 + i\pi/2$.
c) $e^{-\pi/2}$.

2 a) First use the identity $(\cos(\theta) + i\sin(\theta))^2 = (e^{i\theta})^2 = e^{i2\theta} == \cos(2\theta) + i\sin(2\theta)$ to get the double angle formulas for $\cos(2\theta)$ and $\sin(2\theta)$.
b) Express $\cos(4\theta)$ and $\sin(4\theta)$ as polynomials in $\cos(\theta), \sin(\theta)$.

3 a) Find all the eigenvalues of the matrix $A = 3Q = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$ in the complex plane.
b) Verify that if λ is an eigenvalue of Q, then $\vec{v} = \begin{bmatrix} \lambda^3 \\ \lambda^2 \\ \lambda \\ 1 \end{bmatrix}$ is an eigenvector of A.

Solution:
a) $\lambda^4 = 2$ has the solutions $\lambda_k = 2e^{2\pi ik/4}$ which are $\{2, -2, 2i, -2i\}$.

b) Just compute $Av = \begin{bmatrix} 1 \\ \lambda^3 \\ \lambda^2 \\ \lambda \end{bmatrix} = \begin{bmatrix} \lambda^4 \\ \lambda^3 \\ \lambda^2 \\ \lambda \end{bmatrix} = \lambda v$ as $\lambda^4 = 1$.

4 Find the eigenvalues/eigenvectors of $A = \begin{bmatrix} 7 & 4 & 2 & 0 & 0 & 1 \\ 1 & 7 & 4 & 2 & 0 & 0 \\ 0 & 1 & 7 & 4 & 2 & 0 \\ 0 & 0 & 1 & 7 & 4 & 2 \\ 2 & 0 & 0 & 1 & 7 & 4 \\ 4 & 2 & 0 & 0 & 1 & 7 \end{bmatrix}$. Hint: write $A = 7I_6 + 4Q + 2Q^2 + Q^{-1}$?

Solution:
The matrix Q has has $\lambda^6 - 1$ as the characteristic polynomial. It has the roots $\lambda_k = e^{2\pi ik/6}$ with $k = 0, ..., 5$. The eigenvectors are $v_k = \begin{bmatrix} \lambda^5 \\ \lambda^4 \\ \lambda^3 \\ \lambda^2 \\ \lambda \\ 1 \end{bmatrix}$. Now, A has the eigenvalues $7+4\lambda_k+2\lambda_k^2+1/\lambda_k$ and the same eigenvectors.

5 The matrix $A = \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.8 & 0.3 & 0.3 \\ 0.1 & 0.5 & 0.0 \end{bmatrix}$ is called a Markov matrix: in every column the entries add up to 1, so each column can be interpreted as a probability distribution.

a) Verify that A^T has the eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ with eigenvalue 1.

b) Why is 1 an other eigenvalue of A? Find all eigenvalues.

c) Find all eigenvectors of A.

Solution:
The eigenvalues are $1, (-3+4i)/10, (-3-4i)/10$. The eigenvectors are $[55, 83, 47]^T, [-1-2I, -1+2I, 2]^T, [-1+2I, -1-2I, 2]^T$.

Complex eigenvalues

Complex numbers are of the form $z = a + ib$. Add and multiply as real numbers, keeping in mind $i^2 = -1$. Euler found $\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)}$ which for $\theta = \pi$ gives $\boxed{e^{i\pi} + 1 = 0}$ (the most beautiful formula in whole of mathematics), as it combines $e, \pi, i, 1$ and 0. As e is part of analysis and π is part of geometry and 0 is the additive neutral element and 1 the multiplicative neutral element, this identity combines analysis, geometry and algebra. The Euler identity leads to **de Moivre formulas** like $(\cos(\theta)+i\sin(\theta))^3 = (e^{i\theta})^3 = e^{i3\theta} = \cos(3\theta)+i\sin(3\theta)$. So that $\cos^3(\theta) - \cos(\theta)\sin^2(\theta) = \cos(3\theta)$ and $\cos^2(\theta)\sin(\theta) - \sin^3(\theta) = \sin(3\theta)$. Eigenvalues of matrices can become complex as the rotation-dilation matrix $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ shows which has the eigenvalues $a \pm ib$. The fundamental theorem of algebra assures that the sum of the algebraic multiplicities of all eigenvalues of a $n \times n$ matrix is equal to n.