Homework 19: Eigenspaces

This homework is due on Friday, March 23, respectively on Tuesday, March 27, 2018.

1 Find all the eigenvalues for the matrix $A = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & 0 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ -1 & 0 & -1 & 3 & -1 \\ -1 & -1 & 0 & -1 & 3 \end{bmatrix}$.

What are the algebraic and geometric multiplicities? As a hint, we tell you that the eigenvectors are $[-2,0,1,0,1]^T, [-2,1,0,1,0]^T, [0,0,-1,0,1]^T,$ $[0,-1,0,1,0]^T, [1,1,1,1,1]^T$. Now find the characteristic polynomial of A.

2 Assume that a 2×2 matrix has trace 9 and determinant 14. Find its eigenvalues and find a non-diagonal matrix which realizes the situation.

3 a) Verify that for any $n \times n$ matrix A, the matrix A and A^T have the same eigenvalues.
b) Assume A is invertible. What is the relation between the eigenvalues of A and A^{-1}?

4 This is a classic problem from Otto Bretscher. The vector $A^n b$ gives pollution levels in the Silvaplana, Sils and St Moritz lake n weeks after an oil spill. The matrix is $A = \begin{bmatrix} 0.7 & 0 & 0 \\ 0.1 & 0.6 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix}$

and $b = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$ is the initial pollution level. Find a closed form solution for the pollution after n weeks.

5 a) Find all the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 6 & 6 & 0 \\ 1 & 14 & 36 & 24 \end{bmatrix} .$$

b) Find the eigenvectors of A^3, where A is the previous matrix.
c) Find the eigenvectors of $(A^T)^{-1}$, where A is the previous matrix.

Eigenspaces

A nonzero vector $\vec{v}$ is called an **eigenvector**, if $Av = \lambda v$ for some λ. The set of eigenvectors is called the **eigenspace** E_λ. It is the kernel of $A - \lambda I_n$. The dimension of the eigenspace is called the **geometric multiplicity** of λ. There is a general result which tells that the geometric multiplicity of λ is always smaller or equal to the algebraic multiplicity.
Recall that A is similar to B if there exists an invertible S such that $B = S^{-1}AS$. If A and B are similar, then they have the same characteristic polynomial, the same eigenvalues and algebraic multiplicities as well as the same geometric multiplicities. Similar matrices also have the same trace $\text{tr}(A) = \lambda_1 + ... + \lambda_n$ as well as determinant $\det(A) = \lambda_1 \cdots \lambda_n$. These formulas hold in general if we allow the eigenvalues λ_i to be complex. More on complex eigenvalues next week.