Homework 6: Matrix Algebra

This homework is due on Friday, February 9, respectively on Tuesday February 13, 2018.

1 For each pair of matrices A and B, compute both AB and BA

a) $A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 \\ -1 & 3 \end{bmatrix}$.

b) $A = \begin{bmatrix} 2 & 3 & 4 \\ 2 & -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 \\ 2 & 3 \\ 6 & 11 \end{bmatrix}$.

Solution:

a) $AB = \begin{bmatrix} 1 & 13 \\ 0 & 16 \end{bmatrix}$, $BA = \begin{bmatrix} 8 & 14 \\ 4 & 9 \end{bmatrix}$.

b) $AB = \begin{bmatrix} 38 & 57 \\ 6 & 1 \end{bmatrix}$.

$BA = \begin{bmatrix} 12 & 10 & 16 \\ 10 & 3 & 8 \\ 34 & 7 & 24 \end{bmatrix}$.

2 a) Find a 2×2 matrix A with no 0 or 2 entries such that $A^2 = 0$.

b) Can you find a 3×3 matrix A with entries $-2, 1$ such that $A^2 = 0$? (You can search computer assisted).

c) Can you find a 4 times 4 matrix with entries $-1, 3$ such that $A^2 = 0$?

```
(* See what this does and modify *)
Do[A=Table[RandomChoice[{-1,1}],{4},{4}];
  If[Max[Abs[A.A]]==0,Print[A]],{10000}]
```

Solution:

a) $A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ works (note that this is because the rows are orthogonal to both columns). b) An example is

$$\begin{bmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

c) An example is

$$\begin{bmatrix} -1 & -1 & -1 & -1 \\ 3 & 3 & 3 & -1 \\ -1 & 3 & 3 & -1 \\ 3 & 3 & -1 & 3 \end{bmatrix}$$

3 a) Find the inverse of the matrix A made from the first 4 rows of Pascal's triangle. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$.

b) The following $0-1$ matrix B has the property that the inverse is again an integer matrix: $B = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$. Find the inverse.

Solution:

a) $A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$.

b) $B^{-1} = \begin{bmatrix} -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix}$.

4 a) Assume $A^8 = A \cdot A \cdot A \cdot A \cdot A \cdot A \cdot A \cdot A$ is the identity matrix. Can you find a simple formula in terms of A which gives A^{-1}?
b) Find a transformation in the plane with A^4 not being the identity such that $A^8 = 1$. What is A^{-1}?

Solution:
a) It is A^7, because $A(A^7) = (A^7)A = 1$.
b) Rotation by $2\pi/8$. We can write down the matrix explicitly: $A = \begin{bmatrix} \cos(2\pi/8) & -\sin(2\pi/8) \\ \sin(2\pi/8) & \cos(2\pi/8) \end{bmatrix}$. Because of how rotations work, A^6 is rotation by $12\pi/7$. This means that $A^{-1} = \begin{bmatrix} \cos(12\pi/8) & -\sin(12\pi/8) \\ \sin(12\pi/8) & \cos(12\pi/8) \end{bmatrix} = \begin{bmatrix} \cos(2\pi/8) & \sin(2\pi/8) \\ -\sin(2\pi/8) & \cos(2\pi/8) \end{bmatrix}$.

5 a) Assume A is small enough so that $B = 1 + A + A^2 + A^3 + ...$ converges. Verify that B is the inverse of $1 - A$. (Leontief).
b) Use Mathematica to plot A^{-1} for the 100 x 100 matrices defined by $A_{nm} = n^2 + 1.1m$.
c) Use Mathematica to plot A^{-1} for the 100 x 100 matrix $A_{n,m} =$

$gcd(n, m)$, the greatest common divisor. (If you can explain this pattern, this might be a research paper. It seems unexplored).

Solution:

a) By the distributive property, we have that $(1 - A)B = 1B - AB = B - AB = B - A(1 + A + A^2 + \ldots) = B - (A + A^2 + A^3 + \ldots) = B - (B - 1) = 1$.

b) Use the code below. We see strange stripes.

c) We see rays.

```
MatrixPlot[Table[n^2+1.1 m,{n,300},{m,300}]];
```

Matrix Algebra

Matrices can be added, multiplied with a scalar. One can also form the product of two matrices $A \cdot B$ as well as the inverse matrix A^{-1} if the matrix is invertible. These operations constitute the **matrix algebra**. It behaves like the algebra of real numbers but the multiplication is no more commutative in general. Besides the matrix 0 where all entries are zero there are other matrices which are not invertible. We write 1 for the identity matrix which has 1 in the diagonal and 0 everywhere else. Now $A1 = A$.

```
A={{5,2},{3,4}};
Inverse[A]+MatrixPower[A,7] + IdentityMatrix[2]
```