

Homework 3: The number of solutions

This homework is due on Friday, February 2, respectively on Tuesday February 6, 2018.

1 Given $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \end{bmatrix}$. For each of the vectors \vec{b} given below,

determine whether the system $A\vec{x} = \vec{b}$ has 0, 1 or ∞ many solutions.

a) $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ b) $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}$ c) $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ d) $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ e) $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

2 Consider the set X of all 2×2 matrices with matrix entries 1 or 2. The probability of a set of matrices Y with some property is the number of matrices in Y divided by the number of matrices in X .

- a) What is the probability that the rank of the matrix is 0?
- b) What is the probability that the rank of the matrix is 1?
- c) What is the probability that the rank of the matrix is 2?

3 As in the previous problem, now also the 2-vector b takes randomly the values 1,2, we can look at all the possible equations $Ax = b$, where A, b are obtained with 1 or 2 entries. The probability space has now 64 elements.

- a) What is the probability that the system has a unique solution?
- b) What is the probability that the system has no solution?
- c) What is the probability that the system has infinitely many solutions?

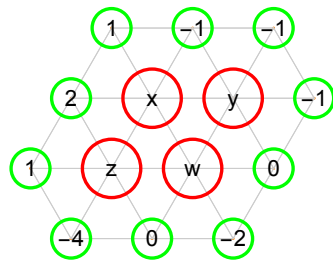
- 4 Build your own system of equations for three variables or state that there is none. Your system has to have the form $a_{11}x + a_{12}y + a_{13}z = b_1$, $a_{21}x + a_{22}y + a_{23}z = b_2$, $a_{31}x + a_{32}y + a_{33}z = b_3$ with all a_{ij} nonzero.
- An example with exactly one solution.
 - An example with no solutions.
 - An example where the solution is a plane.
 - An example where the solution is a line.
 - An example where the solution space is three dimensional.

- 5 In a herb garden, the soil has the property that at any given point the humidity is the sum of the neighboring humidities. Samples are taken on a hexagonal grid on 14 spots. The humidity at the four locations

x, y, z, w is unknown. Solve the equations

$$\left\{ \begin{array}{l} x = y+z+w+2 \\ y = x+w-3 \\ z = x+w-1 \\ w = x+y+z-2 \end{array} \right. \text{ using}$$

row reduction.



Main properties

A system which has a solution is called **consistent**. Otherwise it is called **inconsistent**.

We have a unique solution to $A\vec{x} = \vec{b}$ if and only if $\text{rref}(A)$ has a leading 1 in every column and the system is consistent. We have no solution if and only if $\text{rref}(A|b)$ has a leading 1 in the last column. In all other cases we have infinitely many solutions.