MWF 9 Oliver Knill
MWF 10 Jeremy Hahn
MWF 10 Hunter Spink
MWF 11 Matt Demers
MWF 11 Yu-Wen Hsu
MWF 11 Ben Knudsen
MWF 11 Sander Kupers
MWF 12 Hakim Walker
TTH 10 Ana Balibanu
TTH 10 Morgan Opie
TTH 10 Rosalie Belanger-Rioux
TTH 11:30 Philip Engel
TTH 11:30 Alison Miller

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

Problem 1) (20 points) True or False? No justifications are needed.

1) T F If A is a symmetric matrix such that $A^5 = 0$, then $A = 0$.

Solution:
Since the matrix is symmetric, we can diagonalize it. For diagonal matrices $B = S^{-1}AS$, the statement is true. Therefore also for A.

2) T F If A and B are 3×3 symmetric matrices, then AB is symmetric.

Solution:
$(AB)^T = B^T A^T = BA$ is not equal to AB in general because A and B do not need to commute.

3) T F The solutions of $f'''(x) + f''(x) + f(x) = \sin(x)$ form a linear subspace of all smooth functions.

Solution:
The homogenous equation has a three dimensional solution set. The general solution is the sum of a particular solution and a homogeneous solution. This is not a linear space.

4) T F The initial value problem $f'''(x)+f''(x)+f(x) = \sin(x)$, $f(0) = 0$, $f'(0) = 0$ has exactly one solution.

Solution:
The solution space is one dimensional. Without any initial conditions, we have a three dimensional solution set. The initial condition $f(0) = 0$ fixes one dimension, the other initial condition an other one. If we would also prescribe $f''(0)$, then the differential equation would have exactly one solution.

5) T F Every real 3×3 matrix having $\lambda = 1 + i$ as an eigenvalue is diagonalizable over the complex numbers.

Solution:
Because the matrix A is real it has an other eigenvalue $\overline{\lambda} = 1 - i$. Since a polynomial of degree 3 has at least one real root, we have additionally a real eigenvalue. So, the matrix has 3 different eigenvalues. It is diagonalizable because any matrix with different eigenvalues is diagonalizable.

6) T F If A is a nonzero diagonalizable 4×4 matrix, then A^4 is nonzero.

Solution:
Because A is diagonalizable and nonzero, it has at least one nonzero eigenvalue λ with eigenvector v. Because $A^n v = \lambda^n v \neq 0$, A^n can not be zero for all n. Especially not for $n = 4$.

7) T F There exists a real 2×2 matrix A such that $A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Solution:
Take a rotation-dilation matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

8) T F There exist invertible 2×2 matrices A and B such that $\det(A + B) = \det(A) + \det(B)$.

Solution:
For rotation dilation matrices $\det(A) = a^2 + b^2$ is the square of the length of the associated vector (a, b) respectivly the complex number $z = a + ib$. Take two orthogonal vectors. Pythagoras will assure that the determinant formula $\det(A + B) = \det(A) + \det(B)$ is true in that case. Note that for general matrices, this formula is wrong.

9) T F The kernel of the differential operator D^{100} on $C^\infty(\mathbb{R})$ has dimension 100.

Solution:
The kernel consists of all polynomials of degree 99.

10) T F $Tf(x) = \sin(x)f(x) + f(0) + \int_{-1}^{x} f(y)\ dy$ is a linear transformation on $C^\infty(\mathbb{R})$.

Solution:
Check $T(f + g) = T(f) + T(g), T(\lambda f) = \lambda T(f), T(0) = 0$.

11) T F If $S^{-1}AS = B$, then $\mathrm{tr}(A)/\mathrm{tr}(B) = \det(A)/\det(B)$.

Solution:
Indeed, for similar matrices, both the trace and the determinant agree, so that both the left and right hand side is 1.

12) T (highlighted) | F — If a 3×3 matrix A is invertible, then its rows form a basis of $\mathbb{R}^3$.

Solution:
If A is invertible, so is A^T. The columns of A^T which are the rows of A form a basis.

13) T | F (highlighted) — A 4×4 orthogonal matrix has always a real eigenvalue.

Solution:
Build a partitioned matrix which has two 2×2 rotation matrices in the diagonal and zero matrices in the side diagonal: $A = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 & 0 \\ 0 & 0 & \cos(\beta) & -\sin(\beta) \\ 0 & 0 & -\sin(\beta) & \cos(\beta) \end{bmatrix}$.

14) T (highlighted) | F — If A is orthogonal and B satisfies $B^2 = I_n$ then AB has determinant 1 or -1.

Solution:
The matrix A has determinent $+1$ or -1 because $AA^T = 1_n$. Also B determinant 1 or -1. Therefore, the product has determinant 1 or -1.

15) T | F (highlighted) — If $\frac{d}{dt}\vec{x} = A\vec{x}$ has an asymptotically stable origin then $\frac{d}{dt}\vec{x} = -A\vec{x}$ has an asymptotically stable origin.

Solution:
The real part of the eigenvalues of A are negative. Therefore, the real part of the eigenvalues of $-A$ are positive.

16) T (highlighted) | F — If $\frac{d}{dt}x = Ax$ has an asymptotically stable origin, then the differential equation $\frac{d}{dt}x = Ax + (x \cdot x)x$ has an asymptotically stable origin.

Solution:
Write out the system: $\dot{x} = ax + by + (x^2 + y^2)x, \dot{y} = cx + dy + (x^2 + y^2)y$. The Jacobian matrix at the origin is A. It determines the asymptotic stability also in the nonlinear case.

17) T **F** The transformation on $C^\infty(\mathbb{R})$ given by $T(f)(t) = t + f(t)$ is linear.

Solution:
The constant function 0 is not mapped into 0: $T(0)$ is the function $f(t) = t$.

18) T **F** $\vec{0}$ is a stable equilibrium for the discrete dynamical system $\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$.

Solution:
The trace is 2, the determinant is 2. This is not inside the stability triangle.

19) **T** F If A is an arbitrary 4×4 matrix, then A and A^T are similar.

Solution:
It is true for diagonalizable matrices because the eigenvalues of A and A^T are the same. In general, note that the geometric multiplicities of each eigenvalue of A is the same then the geometric multiplicity of the corresponding eigenvalue of A^T. A linear transformation which matches the corresponding eigenvectors does the trick.

20) **T** F If A is an invertible 4×4 matrix, then the unique least squares solution to $Ax = b$ is $A^{-1}b$.

Solution:
Yes, in that case, the least squares solution is the actual solution. One can also see it formally from the identity $(A^TA)^{-1}A^T = A$.

Total

Problem 2) (10 points)

Match the following objects with the correct description. Every equation matches exactly one description.

a) $\frac{d}{dt}x = 3x - 5y, \frac{d}{dt}y = 2x - 3y$
b) $f_t = f_{xx} + f_{yy}$.
c) $D^2 f(x) + Df(x) - f(x) = \sin(x)$
d) $\frac{d}{dt}x = 3x^3 - 5y, \frac{d}{dt}y = x^2 + y^2 + 2$
e) $x + y = 3, 7x + 3y = 4, 8x + 5y = 10$.
f) $\frac{d}{dt}x + 3x = 0$.

i) An inhomogenous linear ordinary differential equation.
ii) A partial differential equation.
iii) A linear ordinary differential equation with two variables.
iv) A homogeneous one-dimensional first order linear ordinary differential equation.
v) A nonlinear ordinary differential equation.
vi) A system of linear equations.

Solution:
a) iii)
b) ii)
c) i)
d) v)
e) vi)
f) iv)

Problem 3) (10 points)

Define $A = \begin{bmatrix} 1 & -2 & 3 & -4 \\ -5 & 6 & -7 & 8 \\ 9 & -10 & 11 & -12 \end{bmatrix}$.

a) Find $\text{rref}(A)$, the reduced row echelon form of A.
b) Find a bases for $\ker(A)$ and $\text{im}(A)$.
c) Find an orthonormal basis for $\ker(A)$.
d) Verify that $\vec{v} \in \ker(A)$, where $\vec{v} = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix}$.
e) Express $\vec{v}$ in terms of your orthonormal basis for $\ker(A)$.

Solution:

a) $\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

b) The first two columns contain leading 1. Therefore, the first two columns $\begin{bmatrix} 1 \\ -5 \\ 9 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 6 \\ -10 \end{bmatrix}$ of A form a basis for the image of A. To get a basis for the kernel, produce free variables s, t for the last two columns. The equations $z = s, w = t, y - 2s + 3t = 0, x - s + 2t = 0$ show that the general kernel element is $(s - 2t, 2s - 3t, s, t)$. Therefore $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$ form a basis for the kernel.

c) Do Gram-Schmidt orthogonalization: $\vec{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} / \sqrt{6}$, $\vec{u}_2 = \vec{v}_2 - (\vec{v}_2, \vec{w}_1)\vec{w}_1 = \begin{bmatrix} -2 \\ -1 \\ 4 \\ 3 \end{bmatrix} /3$. Normalize this to get $\vec{w}_2 = \begin{bmatrix} -2 \\ -1 \\ 4 \\ 3 \end{bmatrix} / \sqrt{30}$.

d) Just check that $A\vec{v} = \vec{0}$.

e) $\vec{v} = (\vec{v} \cdot \vec{w}_1)\vec{w}_1 + (\vec{v} \cdot \vec{w}_2)\vec{w}_2 = 4\sqrt{2/3}\vec{w}_1 + (7\sqrt{2/15} + \sqrt{6/5})\vec{w}_2$.

Problem 4) (10 points)

Find all solutions to the differential equation

$$f''(t) - 2f'(t) + f(t) = 4e^{3t} .$$

Find the unique solution given the initial conditions $f(0) = 1$ and $f'(0) = 1$.

Solution:

First find a solution to the homogeneous equation: $(D^2 - 2D + 1)f = 0$. Because $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$, the homogeneous solution is a linear combination of the functions e^t and te^t. $\boxed{f(t) = ae^t + bte^t}$.

To get a special solution, we try $f = ce^{3t}$. Plugging this into the differential equation gives $(9c - 6c + c)e^{3t}4e^{3t}$ so that $c = 1$. The general solution is $\boxed{f(t) = ae^t + bte^t + e^{3t}}$

The conditions $f(0) = 1$ and $f'(0) = 1$ fixes the constants: $a + 1 = 1, a + b + 3 = 1$: $a = 0$ and $b = -2$. The unique solution is $\boxed{f(t) = -2te^t + e^{3t}}$.

Problem 5) (10 points)

a) Let $f(x)$ be the function which is 1 on $[\pi/3, 2\pi/3]$ and zero elsewhere on the interval $[0, \pi]$. Write f as a Fourier sin-series.

b) Find the solution to the heat equation $u_t = \mu u_{xx}$ with $u(x, 0) = f(x)$, where μ is a parameter.

c) Find the solution to the wave equation $u_{tt} = c^2 u_{xx}$ with $u(x, 0) = f(x)$ and for which $u_t(x, 0) = 0$ holds for all x. The constant c is a parameter.

Solution:
a) We continue f as an odd function on $[-\pi, \pi]$. The Fourier coefficients are

$$b_n = \frac{2}{\pi} \int_{\pi/3}^{2\pi/3} f(x) \sin(nx)\, dx = \frac{2}{\pi} \int_{\pi/3}^{2\pi/3} \sin(nx)\, dx = \frac{2}{\pi}(\cos(n\pi/3) - \cos(n2\pi/3))/n\ .$$

It is ok to leave the result like this. It could be simplified more because one has $b_{6n-4} = -1$ and $b_{6n-2} = 1$, all other coefficients vanish.
$f(x) = \sum_n b_n \sin(nx)$. b) $u(x, t) = \sum_n b_n e^{-n^2 \mu t} \sin(nx)$.
c) $u(x, t) = \sum_n b_n \cos(nct) \sin(nx)$.

Problem 6) (10 points)

Find a single 3×3 matrix A for which all of the following properties are true.

a) The kernel of A is the line spanned by the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

b) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector for A.

c) $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is in the image of A.

Solution:
There are many solutions to this problem. Because of c), we can put the third vector as a column vector. The puzzle is to complete the matrix $A = \begin{bmatrix} 1 & . & . \\ 2 & . & . \\ -1 & . & . \end{bmatrix}$, so that conditions a,b) hold. Because of a), we know that the sum of the column vectors is the zero vector. We have $A = \begin{bmatrix} 1 & a & -1-a \\ 2 & b & -2-b \\ -1 & c & 1-c \end{bmatrix}$, where we do not know a, b, c. Because of b), we know that there is a λ such that $A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. This means that $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. We have to have $a = c, b = 0$. An example of a solution is to take $a = c = 1, b = 0$: $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 0 & -2 \\ -1 & 1 & 0 \end{bmatrix}$.

Problem 7) (10 points)

a) Find all solutions to the differential equation $(D^2 - 3D + 2)f = 60e^{7x}$.
b) Find all solutions to the differential equation $(D^2 - 2D + 1)f = x$.
c) Find all solutions to the differential equation $(D^2 + 1)f = x^2$.

Solution:
a) We have $(D^2-3D+2) = (D-1)(D-2)$. The general homogeneous solution is ae^x+be^{2x}. A special solution is obtained by trying ce^{7x} which gives $(49c - 21c + 2c)e^{7x} = 60e^{7x}$ or $c = 2$. The general solution is $\boxed{2e^{7x} + ae^x + be^{2x}}$.
b) We have $(D^2 - 2D + 1) = (D - 1)^2$. The general homogeneous solution is $ae^x + bxe^x$. A special solution is obtained by trying $f(x) = ax + b$ for which $(D^2 - 2D + 1)f = -2a + (ax + b) = ax + (-2a + b)$ such that $a = 1, b = 2$. The general solution is of the form $\boxed{2 + x + ae^x + bxe^x}$.
c) The general homogeneous solution is $a\cos(x) + b\sin(x)$. A special inhomogensous solution is obtained by trying $f(x) = ax^2 + bx + c$ for which $(D^2 + 1)f = 2a + ax^2 + bx + c$ in order that this is x^2, we have $a = 1, c = -2, b = 0$. The general solution is of the form $\boxed{x^2 - 2 + a\cos(x) + b\sin(x)}$.

Problem 8) (10 points)

Find the matrix for the rotation in $\mathbb{R}^3$ by 90° about the line spanned by $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, in a clockwise direction as viewed when facing the origin from the point $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$. You get full credit if you leave the result written as a product of matrices or their inverses.

Solution:
We find an orthonormal basis $\mathcal{B}$ for which one vector is in the line: $[2,2,1]/3, [2,-2,0]/\sqrt{8}, [2,2,-8]/\sqrt{72}$ will do. Form the matrix $S = \begin{bmatrix} 2/3 & 2/\sqrt{8} & 2/\sqrt{72} \\ 2/3 & -2/\sqrt{8} & 2/\sqrt{72} \\ 1/3 & 0 & -8/\sqrt{72} \end{bmatrix}$ which brings you from the standard basis to the adapted basis $\mathcal{B}$. The matrix S^{-1} transforms from the new basis to the standard basis. In the standard basis, a rotation around the x axes in the clockwise direction when facing the origin is given by $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$. The matrix, we are looking for is $A = SBS^{-1}$.

Problem 9) (10 points)

a) Find the eigenvalues of the matrix $A = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$.
b) Is $\vec{0}$ a stable equilibrium point for the linear system

$$\frac{d\vec{x}}{dt} = A\vec{x} \ ?$$

c) Describe, how the solution curves of $\frac{d\vec{x}}{dt} = A\vec{x}$ look like.
d) Is $\vec{0}$ a stable equilibrium for the discrete dynamical system $x_{n+1} = Ax_n$?

Solution:
a) The characteristic polynomial is $f_A(\lambda) = \lambda^2 - \lambda + 1/2$. The eigenvalues are $(1+i)/2, (1-i)/2$.
b) The real part of the eigenvalues is positive. The system is unstable.
c) Execept for the initial point 0, all solutions spiral out. When following a solution, we escape to infinity.
d) Yes, each eigenvalue has absolute values $|\lambda| < 1$. The solutions are stable.

Problem 10) (10 points)

Does the system

$$\frac{d\vec{x}}{dt} = B\vec{x}$$

with

$$B = \begin{bmatrix} 0 & -1 & -9 & -9 & -8 \\ 0 & 0 & 0 & -1 & -9 \\ 5 & 0 & 5 & 0 & -5 \\ 1 & 9 & 0 & 0 & 0 \\ 1 & 9 & 9 & 8 & 0 \end{bmatrix}$$

have a stable origin?

Solution:
If the system had a stable origin, then all eigenvalues would have a negative real part. Also the sum of the eigenvalues would have a negative real part. The sum is however equal to 5 because it is the trace of B. So, the system can not have a stable origin.

Problem 11) (10 points)

A 4×4 matrix A is called **symplectic** if $AJA^T = J$, where $J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$.

a) Verify that J itself is symplectic.
b) Show that if A is symplectic, then A is invertible and A^{-1} is symplectic.
c) Check that if both A and B are symplectic, then AB is symplectic.
d) Show that for a symplectic matrix A, one has $\det(A) = 1$ or $\det(A) = -1$.

Solution:
a) The statement follows from $JJ^T = 1$.
b) If A were not invertible, then AJA^T were not invertible. But J is invertible. $AJA^T = J$ is equivalent to $A^{-1}J(A^{-1})^T = J$.
c) $ABJ(AB)^T = ABJB^TA^T = AJA^T = J$.
d) From $\det(A) = \det(A^T)$ and the product formula for the determinant and since $\det(J) = 1$, we have $\det(A)^2 = 1$.
Remark. One can actually show that symplectic matrices always satisfy the equation $\det(A) = 1$. But this is a more difficult task to establish.

Problem 12) (10 points)

Find the ellipse $f(x,y) = ax^2 + by^2 - 1 = 0$ which best fits the data $(2,2), (-1,1), (-1,-1), (2,-1)$.

Solution:
We write down a linear system of equations for the unknowns (a,b) which tells that all the data points are on the ellipse. The least square solution to this system is our best guess.

$$\begin{aligned} 4a + 4b &= 1 \\ a + b &= 1 \\ a + b &= 1 \\ 4a + b &= 1 \end{aligned}$$

which can be written as $Ax = b$ with $b = [1,1,1,1]^T$ and $A = \begin{bmatrix} 4 & 4 \\ 1 & 1 \\ 1 & 1 \\ 4 & 1 \end{bmatrix}$. Calculate $A^TA = \begin{bmatrix} 34 & 22 \\ 22 & 19 \end{bmatrix}$ and $A^Tb = \begin{bmatrix} 10 \\ 7 \end{bmatrix}$ and from that $\begin{bmatrix} a \\ b \end{bmatrix} = (A^TA)^{-1}A^Tb = \begin{bmatrix} 2/9 \\ 1/9 \end{bmatrix}$.
The best ellipse is $2x^2 + y^2 = 9$.

Problem 13) (10 points)

We analyze the nonlinear system of differential equations

$$\begin{aligned} \dot{x} &= x^2 - y^2 + 3 \\ \dot{y} &= -x + 2y - 3 \end{aligned}$$

a) Find the nullclines.
b) There is one equilibrium point. Find it.
c) Find the eigenvalues of the Jacobian at the equilibrium.
d) Is the equilibrium stable? Explain.

Solution:
a) The x-nullclines are hyperbola, the y-nullclines form a line.
b) The point $(1,2)$ is the only equilibrium point.
c) The Jacobian matrix is $\begin{bmatrix} 2x & -2y \\ -1 & 2 \end{bmatrix}$. At the point $(1,2)$ the eigevanlues are 0 and 4.
d) The equilibrium point is unstable.

Problem 14) (10 points)

Find the solution of the partial differential equation

$$u_t = -u_{xxxx} + u_{xx} - u$$

with initial condition $u(x,0) = f(x) = |\sin(x)|$ on the interval $[0,\pi]$.

Solution:
This system can be solved in the same way as the heat equation. We can write $u_t = T(u)$ and the functions $\sin(nx)$ are eigenfunctions of the operator $T = -D^4 + D^2 - 1$. The eigenvalues are $\lambda_n = -n^4 - n^2 - 1$. If $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$, the solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{(-n^4-n^2-1)t} \sin(nx) \ .$$

Note that $|\sin(x)| = \sin(x)$ on $[0,\pi]$ so that this is already the Fourier series. We have $b_1 = 1$ and all other terms are 0. The solution si $u(x,t) = e^{-3t} sin(x)$.