ODE COOKBOOK Math 21b, 2018

$$x' - \lambda x = 0 \qquad x(t) = Ce^{\lambda t}$$

This first order ODE is by far the most important differential equation. A linear system of differential equation $x'(t) = Ax(t)$ reduces to this after diagonalization. We can rewrite the differential equation $x' - \lambda x = 0$ as $(D - \lambda)x = 0$. That is x is in the kernel of $D - \lambda$. An other interpretation is that $\exp(\lambda x)$ is an eigenfunction of D belonging to the eigenvalue λ. This differential equation describes exponential growth or exponential decay.

$$x'' + k^2 x = 0 \qquad x(t) = A\cos(kt) + B\sin(kt)$$

This second order ODE is by far the second most important differential equation. Any linear system of differential equations $x''(t) = Ax(t)$ reduces to this by diagonalization. We can rewrite the differential equation $x'' + k^2 x = 0$ as $(D^2 + k^2)x = 0$. That is, x is in the kernel of $D^2 + k^2$. An other interpretation is that x is an **eigenfunction** of D^2 belonging to the eigenvalue $-k^2$. This differential equation describes oscillations or waves.

OPERATOR METHOD. A general method to find solutions to $p(D)f = g$ is to factor the polynomial $p(D) = (D - \lambda_1) \cdots (D - \lambda_n)x = g$, then invert each factor to get

$$x = (D - \lambda_n)^{-1} \ldots\ldots (D - \lambda_1)^{-1} g$$

where

$$(D - \lambda)^{-1} g = Ce^{\lambda t} + e^{\lambda t} \int_0^t e^{-\lambda s} g(s)\, ds$$

COOKBOOK METHOD. The operator method always works. But it can produce a considerable amount of work. Engineers therefore rely also on cookbook recipes. The solution of an inhomogeneous differential equation $p(D)x = g$ is found by first finding the **homogeneous solution** x_h which is the solution to $p(D)x = 0$. Then a **particular solution** x_p of the system $p(D)x = g$ is found by an educated guess. This method is often much faster but it requires to know the "recipes". Fortunately, it is quite easy: as a rule of thumb: feed in the same class of functions which you see on the right hand side and if the right hand side should contain a function in the kernel of $p(D)$, try with a function multiplied by t. The general solution of the system $p(D)x = g$ is $x = x_h + x_p$.

FINDING THE HOMOGENEOUS SOLUTION. $p(D) = (D - \lambda_1)(D - \lambda_2) = D^2 + bD + c$. The next table covers all cases for homogeneous second order differential equations $x'' + px' + q = 0$.

$\lambda_1 \neq \lambda_2$ real	$C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$
$\lambda_1 = \lambda_2$ real	$C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t}$
$ik = \lambda_1 = -\lambda_2$ imaginary	$C_1 \cos(kt) + C_2 \sin(kt)$
$\lambda_1 = a + ik, \lambda_2 = a - ik$	$C_1 e^{at} \cos(kt) + C_2 e^{at} \sin(kt)$

FINDING AN INHOMOGENEOUS SOLUTION. Inhomogeneous solutions can be found by applying the operator inversions with $C = 0$ or by an educated guess. For $x'' = g(t)$ we just integrate twice, otherwise, check with the following table:

$g(t) = a$ constant	$x(t) = A$ constant
$g(t) = at + b$	$x(t) = At + B$
$g(t) = at^2 + bt + c$	$x(t) = At^2 + Bt + C$
$g(t) = a\cos(bt)$	$x(t) = A\cos(bt) + B\sin(bt)$
$g(t) = a\sin(bt)$	$x(t) = A\cos(bt) + B\sin(bt)$
$g(t) = a\cos(bt)$ with $p(D)g = 0$	$x(t) = At\cos(bt) + Bt\sin(bt)$
$g(t) = a\sin(bt)$ with $p(D)g = 0$	$x(t) = At\cos(bt) + Bt\sin(bt)$
$g(t) = ae^{bt}$	$x(t) = Ae^{bt}$
$g(t) = ae^{bt}$ with $p(D)g = 0$	$x(t) = Ate^{bt}$
$g(t) = q(t)$ polynomial	$x(t) =$ polynomial of same degree

EXAMPLE 1: $$f'' = \cos(5x)$$

This is of the form $D^2 f = g$ and can be solved by inverting D which is integration: integrate a first time to get $Df = C_1 + \sin(5x)/5$. Integrate a second time to get

$$f = C_2 + C_1 t - \cos(5t)/25$$

This is the operator method in the case $\lambda = 0$.

EXAMPLE 2: $$f' - 2f = 2t^2 - 1$$

This homogeneous differential equation $f' - 5f = 0$ is hardwired to our brain. We know its solution is Ce^{2t}. To get a homogeneous solution, try $f(t) = At^2 + Bt + C$. We have to compare coefficients of $f' - 2f = -2At^2 + (2A - 2B)t + B - 2C = 2t^2 - 1$. We see that $A = -1, B = -1, C = 0$. The special solution is $-t^2 - t$. The complete solution is

$$f = -t^2 - t + Ce^{2t}$$

EXAMPLE 3: $$f' - 2f = e^{2t}$$

In this case, the right hand side is in the kernel of the operator $T = D - 2$ in equation $T(f) = g$. The homogeneous solution is the same as in example 2, to find the inhomogeneous solution, try $f(t) = Ate^{2t}$. We get $f' - 2f = Ae^{2t}$ so that $A = 1$. The complete solution is

$$f = te^{2t} + Ce^{2t}$$

EXAMPLE 4: $$f'' - 4f = e^t$$

To find the solution of the homogeneous equation $(D^2 - 4)f = 0$, we factor $(D-2)(D+2)f = 0$ and add solutions of $(D-2)f = 0$ and $(D+2)f = 0$ which gives $C_1 e^{2t} + C_2 e^{-2t}$. To get a special solution, we try Ae^t and get from $f'' - 4f = e^t$ that $A = -1/3$. The complete solution is

$$f = -e^t/3 + C_1 e^{2t} + C_2 e^{-2t}$$

EXAMPLE 5: $$f'' - 4f = e^{2t}$$

The homogeneous solution $C_1 e^{2t} + C_2 e^{-2t}$ is the same as before. To get a special solution, we can not use Ae^{2t} because it is in the kernel of $D^2 - 4$. We try Ate^{2t}, compare coefficients and get

$$f = te^{2t}/4 + C_1 e^{2t} + C_2 e^{-2t}$$

EXAMPLE 6: $f'' + 4f = e^t$

The homogeneous equation is a harmonic oscillator with solution $C_1 \cos(2t) + C_2 \sin(2t)$. To get a special solution, we try Ae^t compare coefficients and get

$$f = e^t/5 + C_1 \cos(2t) + C_2 \sin(2t)$$

EXAMPLE 7: $f'' + 4f = \sin(t)$

The homogeneous solution $C_1 \cos(2t) + C_2 \sin(2t)$ is the same as in the last example. To get a special solution, we try $A \sin(t) + B \cos(t)$ compare coefficients (because we have only even derivatives, we can even try $A \sin(t)$) and get $f = \sin(t)/3 + C_1 \cos(2t) + C_2 \sin(2t)$

EXAMPLE 8: $f'' + 4f = \sin(2t)$

The solution $C_1 \cos(2t) + C_2 \sin(2t)$ is the same as in the last example. To get a special solution, we can not try $A \sin(t)$ because it is in the kernel of the operator. We try $At \sin(2t) + Bt \cos(2t)$ instead and compare coefficients $f = t \sin(2t)/16 - t \cos(2t)/4 + C_1 \cos(2t) + C_2 \sin(2t)$

EXAMPLE 9: $f'' + 8f' + 16f = \sin(5t)$

The homogeneous solution is $C_1 e^{-4t} + C_2 te^{-4t}$. To get a special solution, we try $A \sin(5t) + B \cos(5t)$ compare coefficients and get $f = -40 \cos(5t)/41^2 + -9 \sin(t)/41^2 + C_1 e^{-4t} + C_2 te^{-4t}$

EXAMPLE 10: $f'' + 8f' + 16f = e^{-4t}$

The homogeneous solution is still $C_1 e^{-4t} + C_2 te^{-4t}$. To get a special solution, we can not try e^{-4t} nor te^{-4t} because both are in the kernel. Add an other t and try with $At^2 e^{-4t}$. $f = t^2 e^{-4t}/2 + C_1 e^{-4t} + C_2 te^{-4t}$

EXAMPLE 11: $f'' + f' + f = e^{-4t}$

By factoring $D^2 + D + 1 = (D - (1 + \sqrt{3}i)/2)(D - (1 - \sqrt{3}i)/2)$ we get the homogeneous solution $C_1 e^{-t/2} \cos(\sqrt{3}t/2) + C_2 e^{-t/2} \sin(\sqrt{3}t/2)$. For a special solution, try Ae^{-4t}. Comparing coefficients gives $A = 1/13$. $f = e^{-4t}/13 + C_1 e^{-t/2} \cos(\sqrt{3}t/2) + C_2 e^{-t/2} \sin(\sqrt{3}t/2)$

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