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- Please fill in your name and mark your section.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) (20 points) True or False? No justifications are needed.

1) **T** F The formula for the projection onto the image of a matrix A is $A(A^TA)^{-1}A^T$.

Solution:
It is $A(A^TA)^{-1}A^T$.

2) T **F** Let A and B be two 3×3 matrices. Then A and B are similar if and only if they have the same eigenvalues.

Solution:
The shear is a counter example.

3) T **F** Let A be any $n \times n$ matrix. If $\det(A^TA) = 1$ or $\det(A^TA) = -1$, then A is an orthogonal matrix.

Solution:
The diagonal matrix $\text{diag}(2, 1/2)$ is a counter example.

4) T **F** A matrix which is both symmetric $A^T = A$ and skew-symmetric $A^T = -A$ is orthogonal.

Solution:
This is already false for rotations in the plane.

5) T **F** The trace of a matrix A does not change under row reduction.

Solution:
Every invertible matrix reduces to the identity matrix.

6) **T** F The eigenspace to the eigenvalue 0 of a matrix A does not change under row reduction.

Solution:
While the eigenvectors can change, the kernel does not. The eigenspaces to other eigenvalues do not stay invariant under row reduction in general.

7) | T | F | (T marked)

If A is any matrix of a rotation around a line in space, then $\det(A - I) = 0$.

Solution:
Indeed, a nonzero vector in the axes of rotation is an eigenvector to the eigenvalue 1.

8) | T | F | (F marked)

If A is diagonalizable and B is diagonalizable, then $A + B$ is always diagonalizable.

Solution:
Start with $A = \text{diag}(1/2, 1/3)$ and $B = \begin{bmatrix} 1/2 & 1 \\ 0 & 2/3 \end{bmatrix}$, then $A + B$ is a shear.

9) | T | F | (F marked)

There is a recursion $x_{n+1} = ax_n + bx_{n-1}$ for which $x_n = \sqrt{n}$ for all n.

Solution:
Diagonalization of linear systems shows that x_n can only grow exponentially or linearly or oscillate.

10) | T | F | (F marked)

The product of two reflections at a line in the plane is always a reflection at a line.

Solution:
Take twice the same reflection.

11) | T | F | (F marked)

The characteristic polynomial of A is the same as the characteristic polynomial of A^{-1}.

Solution:
While we have $\det(\lambda - A) = \det(\lambda - A^T)$ we do not have this identity for A^{-1} it is already wrong for $\lambda = 0$ if A has not the determinant 1 or -1.

12) T **F** For any 3×3 matrix, we have $\det(A^4) = \det(A)^{12}$.

Solution:
No, $\det(A^n) = \det(A)^n$.

13) T **F** The determinant of a matrix is equal to the sum of the eigenvalues.

Solution:
The determinant is the product of the eigenvalues, the trace is the sum.

14) T **F** There is a reflection at a line in the plane for which the determinant is equal to 1.

Solution:
The eigenvalues are 1 or -1. The determinant is -1.

15) **T** F The matrix $A = \begin{bmatrix} 100 & 0 \\ 10000 & 1000 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 1000 & 100000 \\ 0 & 100 \end{bmatrix}$.

Solution:
Because both have the same trace and determinant, their eigenvalues are the same. Because the eigenvalues are different they can both be diagonalized to the same diagonal matrix.

16) **T** F If two 2×2 matrices A and B are similar, they have the same trace and determinant.

Solution:
They have the same characteristic polynomial.

17) **T** F If y is in $\text{im}(A)$ then the least square solution to $Ax = y$ is an actual solution to $Ax = y$.

Solution:
Yes, then the quadratic error is zero.

18) T F It is possible that $\operatorname{tr}(A^n)$ and $\operatorname{tr}(A^{-n})$ both grow exponentially.

Solution:
This is the case, if one eigenvalue is smaller than 1 and the other is bigger than 1.

19) T F If a symmetric matrix Q is orthogonal, then Q is diagonal.

Solution:
Take a reflection at a line.

20) T F If $A = QR$ is the QR decomposition of a square matrix, then the eigenvalues of A are the diagonal entries of R.

Solution:
Take the case, where A is an orthogonal matrix. Then $A = A \cdot I$ is the QR decomposition, but the eigenvalues of A are not necessarily all 1.

	Total

Problem 2) (10 points)

a) (4 points) Which of the following matrices are diagonalizable. No justifications are necessary.

1) $\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ ☐ 2) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ ☐

3) $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ ☐ 4) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ ☐

b) (6 points) No justifications are necessary. Check the boxes, for which it is possible to find an example in the class of transformations in three dimensional space indicated to the left: (In this problem we also consider multiplication by zero as a dilation. In other words, also the zero matrix is considered a dilation matrix.)

transformation $T(x) = Ax$	$\det(A) = 1$	$\det(A) = -1$	$\det(A) = 0$	$\det(A) = -2$
orthogonal rotation				
reflection at a line				
projection onto line				
standard shear at x axes				
dilation				
rotation dilation				

Solution:

a) Only the matrix 3) is not diagonalizable. The first matrix has the eigenvalues $0, 0, 3$. The geometric multiplicity of 0 is the nullity of the matrix which is 2. The matrix is diagonalizable. The second matrix has the eigenvalues $2, 3, 1$. Since all eigenvalues are different, the matrix is diagonalizable. The third matrix has three eigenvalues 0 but the geometric multiplicity is 1. The matrix is not diagonalizable. The fourth matrix has the eigenvalues $3, 0, 0$. The dimension of the kernel is 2. The matrix is diagonalizable.

transformation $T(x) = Ax$	$\det(A) = 1$	$\det(A) = -1$	$\det(A) = 0$	$\det(A) = -2$
orthogonal rotation	X			
reflection at a line	X			
projection onto line			X	
standard shear at x axes	X			
dilation	X	X	X	X
rotation dilation	X	X	X	X

Problem 3) (10 points)

Define $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$.

a) (4 points) Find the eigenvectors and the geometric multiplicities of the eigenvalues of A.

b) (3 points) Find the algebraic multiplicities of the eigenvalues.

c) (3 points) Find the characteristic polynomial $f_A(\lambda)$ of A.

Solution:

a) There is an eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ with eigenvalue 21. Because the matrix is not invertible, it has a kernel. Row reduction gives $rref(A) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. There is one leading one, the image is one dimensional and the kernel is 5 dimensional. A basis of this eigenspace is

$$\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} .$$

The geometric multiplicity of the eigenvalue 21 is $\boxed{1}$. The geometric multiplicity of the eigenvalue 0 is $\boxed{5}$.

b) The eigenvalues have algebraic multiplicities $\boxed{\text{1 for the eigenvalue 21}}$ and $\boxed{\text{5 for the eigenvalue 0}}$.

c) The characteristic polynomial is $\boxed{f_A(\lambda) = (-\lambda^5)(21 - \lambda)}$.

Problem 4) (10 points)

Find the function $y = f(x) = ax^2 + bx^3$, which best fits the data

x	y
-1	1
1	3
0	10

Solution:
We have to find the least square solution to the system of equations

$$\begin{aligned} 1a - b &= 1 \\ 1a + b &= 3 \\ 0a + 0b &= 10 \end{aligned}$$

which is in matrix form written as $A\vec{x} = \vec{b}$ with

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 3 \\ 10 \end{bmatrix}.$$

We have $A^T A = 2I_2$ and $A^T b = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. We get the least square solution with the formula

$$x = (A^T A)^{-1} A^T b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

The best fit is the function $\boxed{f(x) = 2x^2 + x^3}$.

Problem 5) (10 points)

a) (4 points) Find all the eigenvalues λ_1, λ_2 and eigenvectors v_1, v_2 of the matrix

$$A = \begin{bmatrix} 7 & 1 \\ 4 & 4 \end{bmatrix}.$$

b) (3 points) Find a formula for the characteristic polynomial $f_{A^n}(\lambda)$ of A^n, where n is a positive integer.

c) (3 points) What is $A^n \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ for a general positive integer n?

Solution:
a) Because the sum of the row entries are constant and equal to 8, the matrix has an eigenvalue $\boxed{8}$ with eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Because the sum of the diagonal elements is 11, which is the sum $\lambda_1 + \lambda_2$ of eigenvalues, we know that the other eigenvector is $\boxed{3}$. To compute the second eigenvector, we find the kernel of $A - 3I$ which is the kernel of

$$A = \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix}$$

which has the kernel spanned by $\boxed{\begin{bmatrix} -1 \\ 4 \end{bmatrix}}$.

b) The matrix A^n has the eigenvalues 8^n and 3^n. The characteristic polynomial is

$$f_A^n(\lambda) = (8^n - \lambda)(3^n - \lambda) .$$

An other possibility to express the characteristic polynomial is to compute the trace and determinant of the power matrix and write $f_A(\lambda) = \lambda^2 - (3^n + 8^n)\lambda + 24^n$.

c) We have to write the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as a linear combination of eigenvectors

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

We have $\boxed{c_1 = 6/5, c_2 = 1/5}$ and get the closed formula

$$A^n v = \frac{6}{5} 8^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{5} 3^n \begin{bmatrix} -1 \\ 4 \end{bmatrix} .$$

Problem 6) (10 points)

Find the point P on the plane V spanned by the two vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

closest to the point $b = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 1 \end{bmatrix}$.

Solution:
We need to find the projection of the vector b onto the space V spanned by the two vectors. We use the formula $P = A(A^TA)^{-1}A^Tb$. We have $A^Tb = \begin{bmatrix} 13 \\ -1 \end{bmatrix}$ and $A^TA = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$.
We can compute the projection matrix

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

and get $\boxed{Pb = \begin{bmatrix} 7/2 \\ 7/2 \\ 3 \\ 3 \end{bmatrix}}$.

One can solve the problem also by first finding an orthonormal basis in V by scaling the two vectors, then directly find $P = QQ^T$, where P is the matrix with the now orthonormal basis as columns.

Problem 7) (10 points)

a) (3 points) Find the determinant of the matrix

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

b) (3 points) Find the determinant of the matrix

$$\begin{bmatrix} 3 & 1 & 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 & 2 & 2 \\ 1 & 0 & 3 & 2 & 2 & 2 \\ 0 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 & 1 & 4 \end{bmatrix}$$

c) (4 points) Find the determinant of

$$\begin{bmatrix} 3 & 2 & 0 & 0 & 0 & 0 \\ 3 & 3 & 2 & 0 & 0 & 0 \\ 3 & 3 & 3 & 2 & 0 & 0 \\ 3 & 3 & 3 & 3 & 2 & 0 \\ 3 & 3 & 3 & 3 & 3 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 \end{bmatrix}$$

Solution:
a) The matrix $A - I$ has the eigenvalue 5 with multiplicity 1 and the eigenvalue 0 with multiplicity 4. The matrix A has the eigenvalue 6 with multiplicity 1 and the eigenvalue 1 with multiplicity 4. The determinant is the product of the eigenvalues which is $\boxed{6}$.
b) This is a partitioned matrix. The determinant of A is the product of the determinants of these matrices which is $(-2) \cdot 60 = \boxed{-120}$.
c) Rowreducing by subtracting each row by the next upper row gives an upper triangular matrix

$$\begin{bmatrix} 3 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

which we get the determinant as the product of the diagonal elements which is $\boxed{3}$.

Problem 8) (10 points)

The recursion

$$x_n = 2x_{n-1} - 2x_{n-2}$$

with $x_0 = 5, x_1 = 2$ can be rewritten as

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = A^n \begin{bmatrix} x_1 \\ x_0 \end{bmatrix}$$

with $A = \begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix}$.
a) (3 points) Find the eigenvalues and eigenvectors of A.

b) (4 points) Find a closed formula for x_n.

c) (3 points) Give an argument which verifies that A is similar to the rotation dilation matrix

$$B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} .$$

Note. You can leave powers of a complex number as they are. A term $(3 + 4i)^n$ for example does not have to be simplified further.

Solution:
a) To find the eigenvalues, find the characteristic polynomial

$$f_A(\lambda) = \lambda^2 - 2\lambda + 2 .$$

It has the roots $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$. The eigenvectors are

$$v_1 = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$$

b) We write the initial condition $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ as a sum of eigenvectors, $c_1 v_1 + c_2 v_2$. Solving the system of equations $2 = c_1(1+I)+c_2(1-I), c_1+c_2 = 5$ gives $c_1 = (5+3i)/2, c+2 = (5-i)/2$. The solution is

$$\begin{bmatrix} x_{n+1} \\ x_n \end{bmatrix} = \frac{5+3i}{2}(1+i)^n \begin{bmatrix} 1+i \\ 1 \end{bmatrix} + \frac{5-3i}{2}(1-i)^n \begin{bmatrix} 1-i \\ 1 \end{bmatrix} .$$

Reading the second entry only gives the formula $\boxed{x_n = \frac{5+3i}{2}(1+i)^n + \frac{5-3i}{2}(1-i)^n}$.
c) Both matrices A and B have the same eigenvalues $1 \pm i$. Because the eigenvalues λ_1, λ_2 are different, the matrices are both diagonalizable and similar to the same diagonal matrix $D = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}$. The two matrices A, B are therefore similar.

Problem 9) (10 points)

Let A be the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} .$$

a) (6 points) Find an orthonormal basis for the image of A.

b) (4 points) Find matrices B and C so that $A = BC$ and B is orthogonal and C is upper triangular.

Solution:
a) do the QR decomposition and keep track of the scaling coefficients and dot products to fill in the blanks in b):

$$w_1 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$w_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} .$$

$$w_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} - \left(\begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ -1\sqrt{6} \end{bmatrix}$$

b)

$$B = \begin{bmatrix} 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix}$$

and

$$C = \begin{bmatrix} \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{3} & 1/\sqrt{3} \\ 0 & 0 & \sqrt{2}/\sqrt{3} \end{bmatrix}$$

Problem 10) (10 points)

a) Find all the eigenvalues λ_1, λ_2 and λ_3 of the matrix $A = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

b) Find a formula for $\mathrm{tr}(A^n)$.

Solution:
a) The characteristic polynomial is

$$f_A(\lambda) = -\lambda^3 + 2\lambda^2 - 3\lambda + 2 \; .$$

eigenvalues are

$$\lambda_1 = (1 + i\sqrt{7})/2, \lambda_2 = (1 - i\sqrt{7})/2, \lambda_3 = 1$$

One can also look at the matrix as a partitioned matrix. There is a 2×2 matrix in the upper left corner which has the eigenvalues λ_1, λ_2.
b) Because $\mathrm{tr}(A^n) = \lambda_1^n + \lambda_2^n + \lambda_3^n$, we have

$$\mathrm{tr}(A^n) = \frac{(1 + i\sqrt{7})^n}{2^n} + \frac{(1 - i\sqrt{7})^n}{2^n} + 1 \; .$$