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- Please fill in your name and mark your section.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) (20 points) True or False? No justifications are needed.

1) T **F** Whenever a matrix A has orthonormal columns, then $A^T A$ is the projection matrix onto the image of A.

Solution:
It is AA^T not $A^T A$.

2) **T** F If $\vec{v}$ is an eigenvector of a 2×2 matrix A, then $\vec{v}$ is an eigenvector of $A + A^{10}$.

Solution:
$Av = \lambda v$ implies $(A + A^{10})\vec{v} = (\lambda + \lambda^{10})\vec{v}$.

3) **T** F If A is similar to B and A is diagonalizable, then B is diagonalizable.

Solution:
If A is diagonalizable, it is similar to a diagonal matrix C. Because it is similar to B, all three matrices A, B, C are pairwise similar.

4) T **F** If a 2×2 matrix A is symmetric and orthogonal, then it is a reflection at a line.

Solution:
$A = -I_2$ is symmetric and orthogonal, but it is not a reflection at a line.

5) T **F** The zero vector $\vec{0}$ is an eigenvector to any eigenvalue λ because $A\vec{0} = \lambda\vec{0}$.

Solution:
The zero vector is by definition not an eigenvector.

6) **T** F The determinant of a 2×2 rotation matrix is always equal to 1.

Solution:
We have $\cos(\alpha)$ in the diagonal and $\pm\sin(\alpha)$ in the side diagonal. The determinant is $\cos^2(\alpha) + \sin^2(\alpha) = 1$.

7) | T | F | If A, B are similar 3×3 matrices, then A and B have the same rank.

Solution:
The dimension of the kernel is the same. Use the dimension formula.

8) | T | F | For any diagonal 2×2 matrix, we have $A^2 - \mathrm{tr}(A)A + \det(A)I_2 = 0$.

Solution:
By the way, this is Sylvester's theorem in a special case.

9) | T | F | There is rotation with an eigenvalue $i = \sqrt{-1}$.

Solution:
It is a rotation by 90 degrees.

10) | T | F | If a matrix A has the QR decomposition $A = QR$, then A is similar to R.

Solution:
Take $R = I$ and $Q = -I$. Then $A = -I$ is not similar to R.

11) | T | F | Every matrix is similar to a diagonal matrix.

Solution:
The shear is a counter example.

12) | T | F | The formula $\det(I_n \det(A)) = \det(A)$ is always true.

Solution:
The correct formula would be $\det(I_n \det(A)) = \det(A)^n$.

13) | T | F | If A, B are similar, then $A + A$ is similar to $B + A$.

Solution:
Here is a counter example: take $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

14) | T | F | If the kernel of a matrix A is the same as the kernel of A^T, then the matrix A is diagonalizable.

Solution:
Take a shear A. Both A and A^T have a trivial kernel but the matrix A is not diagonalizable.

15) | T | F | $p_{A^2}(\lambda) = p_A(\lambda)^2$ for any square matrix A, where $p_A(\lambda)$ is the characteristic polynomial of A.

Solution:
Already the degree of the polynomials does not agree.

16) | T | F | If a diagonalizable matrix satisfies $\det(A) = \det(A^2)$, then the matrix has eigenvalues $1, 0$ or -1.

Solution:
Take a rotation which is not diagonal. It has eigenvalues different from $1, -1$ or 0.

17) | T | F | If $\{v_1, \dots, v_n\}$ is an eigenbasis for a $n \times n$ matrix A, then $\det(A) = \det(B)$, where B has the v_i as column vectors.

Solution:
Multiply one of the eigenbasis vectors by 2. This multiplies the determinant of B but not the determinant of A.

18) | T | F |

The sum of the complex algebraic multiplicities of a $n \times n$ matrix A is equal to n.

Solution:
Yes, this is true by the fundamental theorem of algebra.

19) | T | F |

The rotation matrix $\begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix}$ of a rotation by 30 degrees similar to the rotation matrix of the rotation by -30 degrees.

Solution:
Yes, they are both diagonalizable. An explicit conjugation is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

20) | T | F |

The least square solution x^* of $Ax = b$ has the property that Ax^* is the projection of b onto the image of A.

Solution:
That is how we derived the formula of the orthogonal projection.

Problem 2) (10 points)

a) (4 points) Which of the following matrices have only **real** eigenvalues:

1) $\begin{bmatrix} 2 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ ☐ 2) $\begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ ☐

3) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ ☐ 4) $\begin{bmatrix} 4 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ ☐

b) (6 points) No justifications are necessary. Check the boxes, for which the given matrix has an eigenvalue 1.

a)	☐	The 3×3 matrix of a projection from space onto a line.
b)	☐	The 3×3 matrix of a rotation around a line.
c)	☐	The 2×2 matrix of a vertical shear in the plane.
d)	☐	The 3×3 matrix of a reflection at a plane in space.
e)	☐	The 2×2 matrix of a rotation in the plane by 90 degrees.
f)	☐	The 3×3 matrix of the identity transformation in space.
g)	☐	The matrix $A^T A$, where A is a 3×2 matrix with orthonormal columns.
h)	☐	The matrix AA^T, where A is a 3×2 matrix with orthonormal columns.

Solution:

a) The answer is $\boxed{1),3)}$. Here are more details:

a1) The eigenvalues are $2, 1, 2$

a2) The eigenvalues are $2, 2+i, 2-i$.

a3) The eigenvalues are $0, 0, 3$.

a4) The eigenvalues are $4, i, -i$.

b) Only the rotation in the plane by 90 degrees has no eigenvalue 1. All the other transformations have an eigenvalue 1.

Problem 3) (10 points)

Each of the following 5 statements is either true or false. For every statement, you find 4 arguments, which either confirm or dispute the claim. In each of the 5 statements, there is **exactly one** of the 4 explanations which gives the correct reason for the statement to be true or false. Check the right box. No further explanations are required here.

a) (2pts) **Any 2×2 matrix A which is both orthogonal and symmetric must be I_2**

- ☐ True: The only matrix similar to the identity matrix is the identity.
- ☐ False: The matrix of a reflection is both orthogonal and symmetric.
- ☐ True: The condition implies $A^2 = I$, which means that $A = I$
- ☐ False: An orthogonal projection is both an orthogonal transformation and symmetric.

b) (2pts) **A 2×2 matrix of rank 1 is always diagonalizable.**

- ☐ True: The eigenspace to the eigenvalue 0 must be one dimensional.
- ☐ False: Such a matrix can have two eigenvalues 1. An example is the shear.
- ☐ True: There is no matrix of rank 1 for which all the eigenvalues are 0.
- ☐ False: If B is the horizontal shear, then $A = B - I_2$ is a counter example.

c) (2pts) **If all eigenvalues of a $n \times n$ matrix A are > 0 then $B = A + 100 I_n$ is invertible.**

- ☐ True: The eigenvalues of B are greater than 100.
- ☐ False: It is possible that B has an eigenvalue 0.
- ☐ True: The eigenvectors of B are the same as the eigenvectors of A.
- ☐ False: A horizontal shear A has positive eigenvalues but $A + 100 I_2$ is not invertible.

d) (2pts) **The determinant of a rotation in R^2 is 1.**

☐ True: The matrix of a rotation is an orthogonal matrix.
☐ False: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a counter example .
☐ True: The matrix of a rotation in the plane is $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where $a^2 + b^2 = 1$.
☐ False: Because the matrix of a rotation has complex eigenvalues in general.

e) (2pts) **The horizontal shear** $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ **is similar to a reflection at a line.**

☐ True: Both the shear and the reflection have rank 2 and are therefore invertible.
☐ False: The determinant of a shear is 1 while the determinant of a reflection is -1.
☐ True: Both the shear and a reflection are diagonalizable with eigenvalues 1 or -1.
☐ False: Any reflection is a symmetric matrix while the shear is an orthogonal matrix.

Solution:
a) 2)
b) 4)
c) 1)
d) 3)
e) 2)

Problem 4) (10 points)

Define $A = \begin{bmatrix} 1 & 2 & 3 & 3 & 2 & 1 \\ 2 & 4 & 6 & 6 & 4 & 2 \\ 3 & 6 & 9 & 9 & 6 & 3 \\ 3 & 6 & 9 & 9 & 6 & 3 \\ 2 & 4 & 6 & 6 & 4 & 2 \\ 1 & 2 & 3 & 3 & 2 & 1 \end{bmatrix}$.

a) (6 points) Find the eigenvalues, eigenvectors and the geometric multiplicities of the eigenvalues of A.

b) (2 points) Is A diagonalizable? If yes, write down the diagonal matrix B such that

$$B = S^{-1}AS .$$

c) (2 points) Find the characteristic polynomial $f_A(\lambda)$ of A.

Solution:
a) The matrix has a 5 dimensional kernel as you can see if you rowreduce the matrix. The eigenvectors to the eigenvalue 0 can be read off after doing row reduction of A:

$$rref(A) = \begin{bmatrix} 1 & 2 & 3 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

There are 5 eigenvalues 0. There is also an eigenvalue 28 as you can see by inspecting the trace, which is the sum of the eigenvalues. The eigenvectors to the eigenvalues 0 are

$$\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

The eigenvector to the eigenvalue 28 can be computed by row reducing the matrix $A-28I_6$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

which has the kernel spanned by $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \\ 2 \\ 1 \end{bmatrix}$. An other way to see the eigenvector is to note that the eigenvector has to be the image of the transformation and because all column vectors are parallel to the vector $\vec{v}_6$, it must be one of these column vectors.
b) The matrix is diagonalizable and therefore similar to the diagonal matrix which contains the eigenvalues in the diagonal:

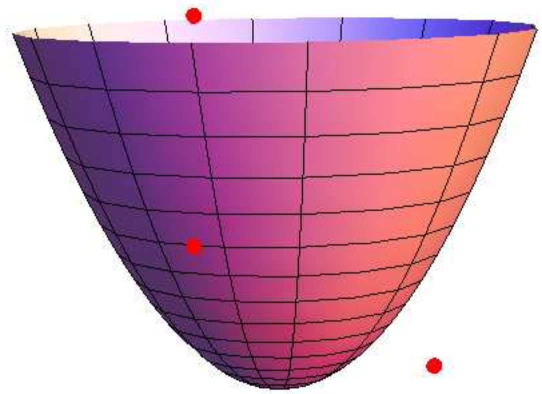
$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 28 \end{bmatrix} .$$

c) We know all the eigenvalues λ_i. The characteristic polynomial is $f_A(\lambda) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$. Therefore, $f_A(\lambda) = \boxed{(-\lambda)^5(28-\lambda)}$.

Problem 5) (10 points)

Which paraboloid $ax^2 + by^2 = z$ best fits the data

x	y	z
0	1	2
-1	0	4
1	-1	3



In other words, find the least square solution for the system of equations for the unknowns a, b which aims to have all data points on the paraboloid.

Solution:
We have to find the least square solution to the system of equations

$$\begin{aligned} a0 + b1 &= 2 \\ a1 + b0 &= 4 \\ a1 + b1 &= 3 \,. \end{aligned}$$

In matrix form this can be written as $A\vec{x} = \vec{b}$ with

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} .$$

We have $A^T A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $A^T b = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$. We get the least square solution with the formula

$$x = (A^T A)^{-1} A^T b = \begin{bmatrix} 3 \\ 1 \end{bmatrix} .$$

The best fit is the function $\boxed{f(x, y) = 3x^2 + y^2}$ which produces an elliptic paraboloid.

Problem 6) (10 points)

a) (4 points) Find all the eigenvalues λ_1, λ_2 and eigenvectors v_1, v_2 of the matrix

$$A = \begin{bmatrix} 9 & 1 \\ 2 & 8 \end{bmatrix} .$$

b) (6 points) Find a closed form solution for the discrete dynamical system

$$\begin{aligned} x(n+1) &= 9x(n) + y(n) \\ y(n+1) &= 2x(n) + 8y(n) \end{aligned}$$

for which $x(0) = 2$, $y(0) = -1$.

Solution:
a) Because the sum of the row entries are constant and equal to 10, the matrix has an eigenvalue $\boxed{10}$ with eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Because the sum of the diagonal elements is 17, which is the sum $\lambda_1 + \lambda_2$ of eigenvalues, we know that the other eigenvector is $\boxed{7}$. To compute the second eigenvector, we find the kernel of $A - I_2$ which is the kernel of

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} .$$

It is spanned by $\boxed{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}$.
We have now to write the initial condition as a sum of the eigenvectors

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} .$$

Solving this simple system of equations for the unknowns c_1, c_2 gives $c_1 = 1, c_2 = -1$ and so the closed form solution

$$A^n \vec{v} = 1 \cdot 10^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \cdot 7^n \begin{bmatrix} -1 \\ 2 \end{bmatrix} .$$

We can write this as $\boxed{x(n) = 10^n + 7^n, y(n) = 10^n - 2 \cdot 7^n}$.

Problem 7) (10 points)

a) (3 points) Find the determinant of the matrix

$$\begin{bmatrix} 12 & 2 & 2 & 2 & 2 \\ 1 & 11 & 1 & 1 & 1 \\ 1 & 1 & 11 & 1 & 1 \\ 1 & 1 & 1 & 11 & 1 \\ 2 & 2 & 2 & 2 & 12 \end{bmatrix}$$

b) (3 points) Find the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 7 & 4 & 1 & 2 \\ 2 & 5 & 8 & 0 & 4 & 1 \\ 3 & 6 & 9 & 0 & 0 & 4 \end{bmatrix}$$

c) (4 points) Find the determinant of

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 1 & 1 & 1 \\ 0 & 0 & 4 & 1 & 1 & 1 \\ 0 & 0 & 4 & 5 & 2 & 2 \\ 0 & 0 & 4 & 1 & 6 & 3 \\ 0 & 0 & 4 & 1 & 1 & 7 \end{bmatrix}$$

Solution:
a) $A - 10I_5$ has eigenvalues $0, 0, 0, 0, 7$ (use the trace to get the last one) so that A has eigenvalues $10, 10, 10, 10, 17$. The determinant is the product of these eigenvalues, which is $\boxed{170'000}$.
b) This is a partitioned matrix. $\det(A) = \boxed{8 \cdot 4^3}$.
c) Also here, first partition the matrix. To get the determinant of the lower matrix, row reduce to get $\boxed{-480}$.

Problem 8) (10 points)

a) (7 points) Find all (possibly complex) eigenvalues of the matrix $A = \begin{bmatrix} 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

b) (3 points) Find the QR decomposition of A.

Solution:

a) The characteristic polynomial is $\lambda^8 - 3^8$. The eigenvalues are therefore the 8'th roots of unity multiplied by 3: $\lambda_k = 3e^{2\pi ik/8} = 3\cos(2\pi k/8) + 3i\sin(2\pi k/8)$, where $k = 0, 1, 2, 3, 4, 5, 6, 7$.

b) The QR decomposition of A is $QR = (A/3)(3I_8)$ because $Q = A/3$ is an orthogonal matrix and $3I_8$ is an upper triangular matrix. The QR decomposition

$$A = \begin{bmatrix} 0&3&0&0&0&0&0&0 \\ 0&0&3&0&0&0&0&0 \\ 0&0&0&3&0&0&0&0 \\ 0&0&0&0&3&0&0&0 \\ 0&0&0&0&0&3&0&0 \\ 0&0&0&0&0&0&3&0 \\ 0&0&0&0&0&0&0&3 \\ 3&0&0&0&0&0&0&0 \end{bmatrix} = \begin{bmatrix} 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0&0 \end{bmatrix} \begin{bmatrix} 3&0&0&0&0&0&0&0 \\ 0&3&0&0&0&0&0&0 \\ 0&0&3&0&0&0&0&0 \\ 0&0&0&3&0&0&0&0 \\ 0&0&0&0&3&0&0&0 \\ 0&0&0&0&0&3&0&0 \\ 0&0&0&0&0&0&3&0 \\ 0&0&0&0&0&0&0&3 \end{bmatrix} = QR \; .$$

is so simple here because the matrix A already had orthogonal columns. We just had to scale them to make them **orthonormal**. (You had seen in a homework that for a matrix with orthogonal columns the QR decomposition has the property that the R matrix is diagonal.

Problem 9) (10 points)

Project the vector $\begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix}$ onto the linear space spanned by the two vectors

$$\{\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}\} \; .$$

Solution:

$A^TA = \begin{bmatrix} 1 & 2 \\ 2 & 16 \end{bmatrix}$. With $b = \begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix}$, we have $A^Tb = \begin{bmatrix} 3 \\ 30 \end{bmatrix}$. Now, $(A^TA)^{-1}A^Tb = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

and $A\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} 3 \\ 4 \\ 4 \\ 4 \end{bmatrix}}$. An alternative method to solve this problem is first to apply Gram-Schmidt to the two vectors to get a 4×2 matrix Q with orthonormal columns and then form QQ^Tb.

Problem 10) (10 points)

a) (5 points) Find an eigenbasis of $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$.

b) (5 points) Do Gram-Schmidt orthogonalization on the basis $\mathcal{B} = \{v_1, v_2, v_3\}$ you just got. Write down the QR decomposition of the matrix S which contains the basis $\mathcal{B}$ as column vectors.

Solution:
a) The eigenvalues are $4, 2, 1$. They are all different. The eigenvectors are

$$\begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} .$$

b) To make a Gram-Schmidt orthogonalization, we use the fact that we can reorder the basis as we want. So, instead of for example $S = \begin{bmatrix} 4 & 2 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, we better take $S = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ which is upper trianguler Gram Schmidt orthonormalization produces the standard basis and the QR decomposition is $S = I_3 S$ because S is already upper triangular. The QR factorization is

$$S = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix} = QR .$$