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- Please fill in your name and mark your section.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) (20 points) True or False? No justifications are needed.

1) T F Whenever a matrix A has orthonormal columns, then A^TA is the projection matrix onto the image of A.

2) T F If $\vec{v}$ is an eigenvector of a 2×2 matrix A, then $\vec{v}$ is an eigenvector of $A + A^{10}$.

3) T F If A is similar to B and A is diagonalizable, then B is diagonalizable.

4) T F If a 2×2 matrix A is symmetric and orthogonal, then it is a reflection at a line.

5) T F The zero vector $\vec{0}$ is an eigenvector to any eigenvalue λ because $A\vec{0} = \lambda\vec{0}$.

6) T F The determinant of a 2×2 rotation matrix is always equal to 1.

7) T F If A, B are similar 3×3 matrices, then A and B have the same rank.

8) T F For any diagonal 2×2 matrix, we have $A^2 - \mathrm{tr}(A)A + \det(A)I_2 = 0$.

9) T F There is rotation with an eigenvalue $i = \sqrt{-1}$.

10) T F If a matrix A has the QR decomposition $A = QR$, then A is similar to R.

11) T F Every matrix is similar to a diagonal matrix.

12) T F The formula $\det(I_n \det(A)) = \det(A)$ is always true.

13) T F If A, B are similar, then $A + A$ is similar to $B + A$.

14) T F If the kernel of a matrix A is the same as the kernel of A^T, then the matrix A is diagonalizable.

15) T F $p_{A^2}(\lambda) = p_A(\lambda)^2$ for any square matrix A, where $p_A(\lambda)$ is the characteristic polynomial of A.

16) T F If a diagonalizable matrix satisfies $\det(A) = \det(A^2)$, then the matrix has eigenvalues $1, 0$ or -1.

17) T F If $\{v_1, \dots, v_n\}$ is an eigenbasis for a $n \times n$ matrix A, then $\det(A) = \det(B)$, where B has the v_i as column vectors.

18) T F The sum of the complex algebraic multiplicities of a $n \times n$ matrix A is equal to n.

19) T F The rotation matrix $\begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix}$ of a rotation by 30 degrees similar to the rotation matrix of the rotation by -30 degrees.

20) T F The least square solution x^* of $Ax = b$ has the property that Ax^* is the projection of b onto the image of A.

Problem 2) (10 points)

a) (4 points) Which of the following matrices have only **real** eigenvalues:

1) $\begin{bmatrix} 2 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ ☐ 2) $\begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ ☐

3) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ ☐ 4) $\begin{bmatrix} 4 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ ☐

b) (6 points) No justifications are necessary. Check the boxes, for which the given matrix has an eigenvalue 1.

a)	☐	The 3×3 matrix of a projection from space onto a line.
b)	☐	The 3×3 matrix of a rotation around a line.
c)	☐	The 2×2 matrix of a vertical shear in the plane.
d)	☐	The 3×3 matrix of a reflection at a plane in space.
e)	☐	The 2×2 matrix of a rotation in the plane by 90 degrees.
f)	☐	The 3×3 matrix of the identity transformation in space.
g)	☐	The matrix $A^T A$, where A is a 3×2 matrix with orthonormal columns.
h)	☐	The matrix AA^T, where A is a 3×2 matrix with orthonormal columns.

Problem 3) (10 points)

Each of the following 5 statements is either true or false. For every statement, you find 4 arguments, which either confirm or dispute the claim. In each of the 5 statements, there is **exactly one** of the 4 explanations which gives the correct reason for the statement to be true or false. Check the right box. No further explanations are required here.

a) (2pts) **Any 2×2 matrix A which is both orthogonal and symmetric must be I_2**

- ☐ True: The only matrix similar to the identity matrix is the identity.
- ☐ False: The matrix of a reflection is both orthogonal and symmetric.
- ☐ True: The condition implies $A^2 = I$, which means that $A = I$
- ☐ False: An orthogonal projection is both an orthogonal transformation and symmetric.

b) (2pts) **A 2×2 matrix of rank 1 is always diagonalizable.**

- ☐ True: The eigenspace to the eigenvalue 0 must be one dimensional.
- ☐ False: Such a matrix can have two eigenvalues 1. An example is the shear.
- ☐ True: There is no matrix of rank 1 for which all the eigenvalues are 0.
- ☐ False: If B is the horizontal shear, then $A = B - I_2$ is a counter example.

c) (2pts) **If all eigenvalues of a $n \times n$ matrix A are > 0 then $B = A + 100I_n$ is invertible.**

- ☐ True: The eigenvalues of B are greater than 100.
- ☐ False: It is possible that B has an eigenvalue 0.
- ☐ True: The eigenvectors of B are the same as the eigenvectors of A.
- ☐ False: A horizontal shear A has positive eigenvalues but $A + 100I_2$ is not invertible.

d) (2pts) **The determinant of a rotation in R^2 is 1.**

- ☐ True: The matrix of a rotation is an orthogonal matrix.
- ☐ False: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a counter example .
- ☐ True: The matrix of a rotation in the plane is $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where $a^2 + b^2 = 1$.
- ☐ False: Because the matrix of a rotation has complex eigenvalues in general.

e) (2pts) **The horizontal shear $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is similar to a reflection at a line.**

- ☐ True: Both the shear and the reflection have rank 2 and are therefore invertible.
- ☐ False: The determinant of a shear is 1 while the determinant of a reflection is -1.
- ☐ True: Both the shear and a reflection are diagonalizable with eigenvalues 1 or -1.
- ☐ False: Any reflection is a symmetric matrix while the shear is an orthogonal matrix.

Problem 4) (10 points)

Define $A = \begin{bmatrix} 1 & 2 & 3 & 3 & 2 & 1 \\ 2 & 4 & 6 & 6 & 4 & 2 \\ 3 & 6 & 9 & 9 & 6 & 3 \\ 3 & 6 & 9 & 9 & 6 & 3 \\ 2 & 4 & 6 & 6 & 4 & 2 \\ 1 & 2 & 3 & 3 & 2 & 1 \end{bmatrix}$.

a) (6 points) Find the eigenvalues, eigenvectors and the geometric multiplicities of the eigenvalues of A.

b) (2 points) Is A diagonalizable? If yes, write down the diagonal matrix B such that

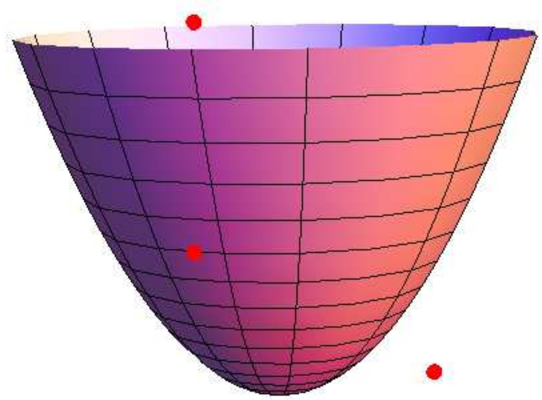
$$B = S^{-1}AS \ .$$

c) (2 points) Find the characteristic polynomial $f_A(\lambda)$ of A.

Problem 5) (10 points)

Which paraboloid $ax^2 + by^2 = z$ best fits the data

x	y	z
0	1	2
-1	0	4
1	-1	3



In other words, find the least square solution for the system of equations for the unknowns a, b which aims to have all data points on the paraboloid.

Problem 6) (10 points)

a) (4 points) Find all the eigenvalues λ_1, λ_2 and eigenvectors v_1, v_2 of the matrix

$$A = \begin{bmatrix} 9 & 1 \\ 2 & 8 \end{bmatrix} .$$

b) (6 points) Find a closed form solution for the discrete dynamical system

$$\begin{aligned} x(n+1) &= 9x(n) + y(n) \\ y(n+1) &= 2x(n) + 8y(n) \end{aligned}$$

for which $x(0) = 2$, $y(0) = -1$.

Problem 7) (10 points)

a) (3 points) Find the determinant of the matrix

$$\begin{bmatrix} 12 & 2 & 2 & 2 & 2 \\ 1 & 11 & 1 & 1 & 1 \\ 1 & 1 & 11 & 1 & 1 \\ 1 & 1 & 1 & 11 & 1 \\ 2 & 2 & 2 & 2 & 12 \end{bmatrix}$$

b) (3 points) Find the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 7 & 4 & 1 & 2 \\ 2 & 5 & 8 & 0 & 4 & 1 \\ 3 & 6 & 9 & 0 & 0 & 4 \end{bmatrix}$$

c) (4 points) Find the determinant of

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 1 & 1 & 1 \\ 0 & 0 & 4 & 1 & 1 & 1 \\ 0 & 0 & 4 & 5 & 2 & 2 \\ 0 & 0 & 4 & 1 & 6 & 3 \\ 0 & 0 & 4 & 1 & 1 & 7 \end{bmatrix}$$

Problem 8) (10 points)

a) (7 points) Find all (possibly complex) eigenvalues of the matrix $A = \begin{bmatrix} 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

b) (3 points) Find the QR decomposition of A.

Problem 9) (10 points)

Project the vector $\begin{bmatrix} 3 \\ 3 \\ 6 \\ 3 \end{bmatrix}$ onto the linear space spanned by the two vectors

$$\{\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}\} .$$

Problem 10) (10 points)

a) (5 points) Find an eigenbasis of $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{bmatrix}$.

b) (5 points) Do Gram-Schmidt orthogonalization on the basis $\mathcal{B} = \{v_1, v_2, v_3\}$ you just got. Write down the QR decomposition of the matrix S which contains the basis $\mathcal{B}$ as column vectors.