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TTH 11:30 Brendan McLellan
TTH 11:30 Krishanu Sankar

- Please fill in your name and mark your section.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) (20 points) True or False? No justifications are needed.

1) T F If A, B are similar $n \times n$ matrices, then A^5 is similar to B^5.

2) T F The columns of a 4×4 orthogonal matrix form an orthonormal basis of R^4.

3) T F If A has orthogonal columns, then AA^T is an orthogonal projection onto the image of A.

4) T F If $\vec{v}$ is an eigenvector of a 3×3 orthogonal matrix A, then it is also an eigenvector of A^T.

5) T F If A is similar to B and A is orthogonal, then B is orthogonal.

6) T F The determinant of a 2×2 projection matrix onto a line is always equal to 0.

7) T F If A, B are similar 5×5 matrices, then A and B have the same nullity.

8) T F If a vertical shear has an eigenvalue 1 of algebraic multiplicity 2 then it is the identity.

9) T F If a matrix A has the QR decomposition $A = QR$ and Q is similar to R, then Q has all eigenvalues 1.

10) T F For any two 2×2 matrices A, B, the characteristic polynomials of AB and BA are the same.

11) T F Every 2×2 matrix with eigenvalues 1 and 2 is similar to a diagonal matrix.

12) T F If the eigenvalues of a matrix A are the same as the eigenvalues of A^{-1}, then A is diagonalizable.

13) T F If $f_A(\lambda)$ is the characteristic polynomial of a $n \times n$ matrix A, then there is a $2n \times 2n$ matrix B such that $f_A^2(\lambda)$ is the characteristic polynomial $f_B(\lambda)$ of B.

14) T F $\det(A + 2B) = \det(A) + 2\det(B)$ is true for any two 5×5 matrices A, B.

15) T F If the projection onto the column space of a matrix A is AA^T, then the columns are an orthonormal basis in the image of A.

16) T F If A and B are 2×2 matrices with the same determinant and the same trace, then A and B are similar.

17) T F If A is the matrix of an orthogonal projection from R^3 onto a plane, then $\det A = 0$.

18) T F If A is a 3×3 matrix then $(A - 5I_3)^T$ has the same eigenvalues as $(A - 5I_3)$.

19) T F The **regular transition** matrix $\begin{bmatrix} 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 0 \\ 0 & 1/3 & 3/4 \end{bmatrix}$ has an eigenvalue of 1.

20) T F If two matrices A, B have the same eigenvalues with the same corresponding algebraic multiplicities, then A and B are similar.

Problem 2) (10 points)

a) (4 points) No justifications are necessary for this problem. Which of the following matrices are diagonalizable over the complex numbers?

1) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ ☐ 2) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 4 \end{bmatrix}$ ☐

3) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ ☐ 4) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ ☐

b) (6 points) No justifications are necessary. Which matrices are orthogonal, which have an eigenvalue -1?

Matrix A	A is orthogonal	A has an eigenvalue -1
$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	☐	☐
$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	☐	☐
$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$	☐	☐

Problem 3) (10 points)

A **Walsh matrix** is a square matrix with dimension 2^n with ± 1 entries and orthogonal columns.

Joseph Walsh was Harvard Professor from 1935 to 1966. His picture hangs in the Math common room.

a) (3 points) What is the determinant and the QR decomposition of the 2×2 Walsh matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} .$$

b) (4 points) Find the QR decomposition of the 4×4 Walsh matrix

$$B = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix} .$$

c) (3 points) Find the determinants of both A and B.

Problem 4) (10 points)

Define the **Harvard matrix** $H = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$.

a) (2 points) Verify that $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ are eigenvectors of H.

b) (4 points) Find all eigenvalues, eigenvectors and the geometric multiplicities of the eigenvalues of H.

c) (2 points) Is H diagonalizable? If yes, write down the diagonal matrix B such that

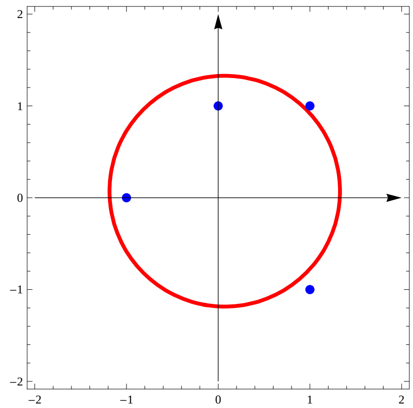
$$B = S^{-1} H S .$$

d) (2 points) Find the characteristic polynomial $f_H(\lambda)$ of H.

Problem 5) (10 points)

Find the circle $a(x^2 + y^2) + b(x + y) = 1$ which best fits the data

x	y
0	1
-1	0
1	-1
1	1



In other words, find the least square solution for the system of equations for the unknowns a, b which aims to have all 4 data points (x_i, y_i) on the circle.

Problem 6) (10 points)

a) (4 points) Find all the eigenvalues λ_1, λ_2 and eigenvectors v_1, v_2 of the matrix

$$A = \begin{bmatrix} 6 & -5 \\ 1 & 0 \end{bmatrix} .$$

b) (6 points) Find a closed form solution for the recursion

$$x(n+1) = 6x(n) - 5x(n-1)$$

for which $x(0) = 3$, $x(1) = 7$.

Problem 7) (10 points)

a) (3 points) Find the determinant of

$$\begin{bmatrix} 0 & 0 & 7 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b) (3 points) Find the determinant of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \end{bmatrix}$$

c) (4 points) Find the determinant of the matrix

$$\begin{bmatrix} 5 & 2 & 3 & 2 & 0 & 0 \\ 6 & 2 & 3 & 2 & 0 & 0 \\ 5 & 3 & 3 & 2 & 0 & 0 \\ 5 & 2 & 3 & 3 & 0 & 0 \\ 2 & 1 & 8 & 3 & 4 & 5 \\ 3 & 6 & 9 & 0 & 3 & 4 \end{bmatrix}$$

Problem 8) (10 points)

Find all (possibly complex) eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

Problem 9) (10 points)

Let $U = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$ define the 6×6 matrix $A = U \cdot U^T$.

a) (4 points) Find a basis for the image of A and find the dimension of the kernel of A.

b) (3 points) What are the eigenvalues of A with their algebraic and geometric multiplicities?

c) (3 points) Find $\det(A + 2I_6)$ and $\operatorname{tr}(A + 2I_6)$.

Problem 10) (10 points)

Let V be the 3-dimensional subspace of $\mathbb{R}^4$ spanned by

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Let $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ be an orthonormal basis of V and A be the matrix $A = \begin{bmatrix} | & | & | \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \\ | & | & | \end{bmatrix}$.

Find $\det(AA^T)$ and $\det(A^T A)$.

Hint. Think before calculating. You don't actually need to compute $\vec{w}_1, \vec{w}_2, \vec{w}_3$, nor do you have to multiply matrices to solve the problem.