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- Please fill in your name and mark your section.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or un-staple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) (20 points) True or False? No justifications are needed.

- 1)  T  F Let  $A$  and  $B$  be two  $2 \times 2$  matrices. Then  $A$  and  $B$  are similar if and only if they have the same trace and determinant.
- 2)  T  F An orthogonal  $n \times n$  matrix satisfies  $AA^T = I_n$ .
- 3)  T  F The eigenvectors of a matrix  $A$  do not change under row reduction.
- 4)  T  F If  $A$  is the matrix of a reflection at a line in space, then  $\det(A^2 - I_3) = 0$ .
- 5)  T  F Every rotation matrix can be diagonalized over the reals.
- 6)  T  F The recursion  $x_{n+1} = x_n + x_{n-1} + x_{n-2} + x_{n-3}$  can be written as  $\vec{v}(n+1) = A\vec{v}_n$  for a  $4 \times 4$  matrix  $A$  and a vector  $\vec{v}$ .
- 7)  T  F There are two rotations in space such that their product is an orthogonal projection onto a plane.
- 8)  T  F The nullity of a square  $A$  is the same as the nullity of  $A^T$ .
- 9)  T  F For any matrix, we have  $\det(A^5/2) = \det(A)^5/2$ .
- 10)  T  F The trace of a matrix is the sum of the eigenvalues
- 11)  T  F There is a reflection for which the determinant is equal to 2.
- 12)  T  F The matrix  $A = \begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$  is similar to  $B = \begin{bmatrix} 4 & 0 \\ 1 & 3 \end{bmatrix}$ .
- 13)  T  F If two  $2 \times 2$  matrices  $A$  and  $B$  have the same trace, determinant and geometric multiplicities of all eigenvalues are 1 then they are similar.
- 14)  T  F If  $A$  has no kernel, then the least square solution of  $Ax = b$  is unique.
- 15)  T  F If  $A^n\vec{v}$  and  $A^{-n}\vec{v}$  both stay bounded for every vector  $\vec{v}$ , then all eigenvalues of  $A$  are located on the unit circle  $|\lambda| = 1$  in the complex plane.
- 16)  T  F If an orthogonal matrix  $Q$  is symmetric, then  $Q$  is diagonal.
- 17)  T  F If  $A = QR$  is the  $QR$  decomposition of a square matrix, then the eigenvalues of  $A$  are the diagonal entries of  $R$ .
- 18)  T  F For every  $2 \times 2$  matrix  $A$ , we have  $A - I_n$  is similar to  $A + I_n$ .
- 19)  T  F If  $A$  is similar to  $B$  then  $A^{-1}$  is similar to  $B^{-1}$ .
- 20)  T  F For any  $2 \times 2$  regular transition matrix, the trace determines the determinant.

Total

Problem 2) (10 points) No justifications are needed.

a) (6 points) Which of the following assertions are true?

true	false	
		$A$ is similar to $B$
		$A$ is similar to $C$
		$A$ is similar to $D$

true	false	
		$B$ is similar to $C$
		$B$ is similar to $D$
		$C$ is similar to $D$

$$A = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$$

b) (4 points) Match the following matrices with sets of eigenvalues. You are told that there is a unique match. It is not always necessary to compute all the eigenvalues to do so.

Enter i),ii),iii) or iv)	The matrix
	$A = \begin{bmatrix} -1 & -2 & 8 \\ -7 & -3 & 19 \\ -3 & -2 & 10 \end{bmatrix}$
	$A = \begin{bmatrix} 5 & -9 & -7 \\ 0 & 5 & 2 \\ 0 & 0 & 6 \end{bmatrix}$
	$A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
	$A = \begin{bmatrix} 13 & 11 & 13 \\ -2 & -1 & -2 \\ -8 & -7 & -8 \end{bmatrix}$

i)  $\{3, 2, 1\}$ .

ii)  $\{1, 0, 3\}$ .

iii)  $\{6, 5, 5\}$ .

iv)  $\{1, i, -i\}$ .

Problem 3) (10 points) No justifications are needed

a) (5 points) Given a unit column vector written as a  $4 \times 1$  matrix  $A = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$ . Fill in the values of the determinants and eigenvalues of the following matrices and state whether they are symmetric

Matrix	Determinant	Eigenvalues	Symmetric? Yes/No
$A^T A$			
$AA^T$			
$A(A^T A)^{-1} A^T$			

b) (5 points) No explanations are necessary for this problem.

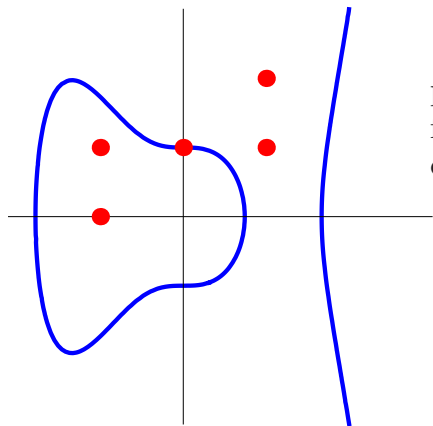
Which matrices are orthogonal, which matrices are anti-symmetric  $A^T = -A$ . Which matrices are projections? Check everything which applies. It is not excluded that you have to check several properties for each matrix.

	orthogonal	antisymmetric	projection	
1)				$A = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$
2)				$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
3)				$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
4)				$A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Problem 4) (10 points)

A curve of the form

$$y^2 = x^5 + ax^3 + b$$



provides examples of **hyper elliptic curves**. Use data fitting to find the best parameters  $(a, b)$  for an elliptic curve given the following points:

$$(x_1, y_1) = (1, 2)$$

$$(x_2, y_2) = (-1, 0)$$

$$(x_3, y_3) = (1, 1)$$

$$(x_4, y_4) = (-1, 1)$$

Problem 5) (10 points)

The sequence  $2, 1, -1, -5, -13, -29, -61, \dots$  satisfies the recursion

$$x(n+1) = 3x(n) - 2x(n-1)$$

with  $x(0) = 2, x(1) = 1$ . Find  $x(n)$ .

**Hint.** As usual, we write the recursion first in the form  $\vec{v}(n+1) = A\vec{v}(n)$ , where  $A$  is a  $2 \times 2$  matrix, where  $\vec{v}(n) = \begin{bmatrix} x(n+1) \\ x(n) \end{bmatrix}$  and where the initial condition is  $\vec{v} = \vec{v}(0) = \begin{bmatrix} x(1) \\ x(0) \end{bmatrix}$ .

Problem 6) (10 points)

a) (4 points) A  $2 \times 2$  matrix  $A$  satisfies  $\text{tr}(A^2) = 5$  and  $\text{tr}(A) = 3$ . Find  $\det(A)$ .

b) (3 points) A  $2 \times 2$  matrix has two parallel columns and  $\text{tr}(A) = 5$ . Find  $\text{tr}(A^2)$ .

c) (3 points) A  $2 \times 2$  matrix  $A$  has  $\det(A) = 5$  and positive integer eigenvalues. What is the trace of  $A$ ?

Problem 7) (10 points)

a) (3 points) Find the determinant of the matrix

$$\begin{bmatrix} 1000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1001 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1002 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 1003 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 1004 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 & 1005 & 5 & 5 \\ 6 & 6 & 6 & 6 & 6 & 6 & 1006 & 6 \\ 7 & 7 & 7 & 7 & 7 & 7 & 7 & 1007 \end{bmatrix}$$

b) (2 points) Find the determinant of the matrix

$$\begin{bmatrix} 3 & 1 & 1 & 2 & 2 & 2 \\ 0 & 3 & 1 & 2 & 2 & 2 \\ 0 & 0 & 3 & 2 & 2 & 2 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 5 & 1 & 4 \end{bmatrix}.$$

c) (3 points) Find the determinant of the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 8 & 9 & 2 & 1 \\ 5 & 2 & 0 & 8 & 9 & 2 & 1 \\ 6 & 4 & 3 & 8 & 9 & 2 & 1 \end{bmatrix}.$$

d) (2 points) Find the determinant of the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Problem 8) (10 points)
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a) (4 points) Find the eigenvalues of the following matrix.

$$A = \begin{bmatrix} 3 & 2 & 3 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

- b) (2 points) Is there a real eigenbasis of  $A$ ?
- c) (4 points) Find the QR decomposition of the matrix  $A$ .

Problem 9) (10 points)

- a) (3 points) Find the eigenvalues of the matrix  $A = \begin{bmatrix} 4 & 0 & 7 \\ 0 & 5 & 0 \\ 2 & 0 & 9 \end{bmatrix}$ .
- b) (4 points) Find the eigenvectors of  $A$ .
- c) (3 points) Find a matrix  $S$  such that  $SB = AS$ , where  $B$  is diagonal.

Problem 10) (10 points)

- a) (2 points) What can you say about the absolute value of an eigenvalue of an orthogonal matrix?
- b) (2 points) What can you say about the eigenvalues of an orthogonal projection?
- c) (2 points) What can you say about the eigenvalues of a shear?
- d) (2 points) What are the eigenvalues of a reflection at a line?
- e) (2 points) You know that an eigenvalues of  $A$  is 3 and an eigenvalues of  $B$  are 5, can you conclude that an eigenvalues of  $AB$  is 10?