| Name: |
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| MWF 9 Oliver Knill |
| MWF 10 Akhil Mathew |
| MWF 10 Ian Shipman |
| MWF 11 Rosalie Belanger-Rioux |
| MWF 11 Stephen Hermes |
| MWF 11 Can Kozcaz |
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| MWF 12 Stephen Hermes |
| MWF 12 Hunter Spink |
| TTH 10 Will Boney |
| TTH 10 Changho Han |
| TTH 11:30 Brendan McLellan |
| TTH 11:30 Krishanu Sankar |

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

| | | |
|---|---|---|
| 1 | | 20 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| 10 | | 10 |
| Total: | | 110 |

Problem 1) TF questions (20 points) No justifications are needed.

1) | T | F | If $T$ is an invertible linear transformation, then its inverse is a linear transformation too.

**Solution:**
The inverse transformation is also given by a matrix, the inverse of the transformation of $A$.

2) | T | F | If $T$ is a non-invertible transformation from $R^2 \to R^2$ then $T$ can not be a linear transformation.

**Solution:**
An extreme case is $T(x) = 0$ which is not invertible, but it is a linear transformation.

3) | T | F | The matrix $A = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$ is an example of a rotation dilation matrix.

**Solution:**
Yes, by definition.

4) | T | F | If a $3 \times 3$ matrix $A$ is invertible, then its rank must be 3.

**Solution:**
Yes, the rank being 3 is equivalent to the kernel be trivial.

5) | T | F | There is a $11 \times 11$ matrix $A$ for which the rank of $A$ is equal to the nullity of $A$.

**Solution:**
The sum of rank and nullity is $n$. If both are the same, then the sum, the number of columns must be even.

6) | T | F | There is a $4 \times 3$ matrix with a 3-dimensional kernel.

**Solution:**
Take the 0 matrix.

7) | T | F | There is a $4 \times 3$ matrix with 4-dimensional image.

**Solution:**
The dimension of the image is the number of leading 1 which is less or equal than the number of columns.

8) | T | F | If a system of equations $Ax = b$ with nonzero $b$ has at least one solution, then the space of solutions is a linear space.

**Solution:**
The solution space is an affine space which passes through the special nonzero solution.

9) | T | F | If a system of equations has only $\vec{0}$ as a solution, then it is called inconsistent.

**Solution:**
Inconsistent means that that there is no solution.

10) | T | F | The map $T(x) = x + 1$ on the real line $R^1$ is an example of a linear transformation.

**Solution:**
It does not map 0 to 0.

11) | T | F | Since the map $T(x) = 1 - x$ satisfies $T(T(x)) = x$, it is a reflection and so a linear map.

**Solution:**
Again, it does not satisfy $T(0) = 0$.

12) | T | F | There is a $31 \times 32$ matrix which has a trivial kernel.

**Solution:**
Trivial kernel means that the rank must be 32 which is not possible as the rank is $\leq 31$.

13) T F The kernel of the $1 \times 3$ matrix $[1\ \ 1\ \ 1]$ is the plane $x + y + z = 0$.

**Solution:**
Yes, this is a reformulation.

14) T F If $A$ is a $2 \times 2$ matrix satisfying $A^2 = 0$, then $(I_2 + A)$ is the inverse of $(I_2 - A)$.

**Solution:**
Just multiply out.

15) T F If a $2 \times 2$ matrix satisfies the relation $A^2 = A$ then $A = 0$ or $A = I_2$.

**Solution:**
It can be a projection onto a one dimensional space.

16) T F The subset of functions $f$ in $C^\infty$ satisfying $f(4) = f'(4) = 0$ is a linear space.

**Solution:**
Check the three properties.

17) T F If $A, B, S$ are invertible $2 \times 2$ matrices satisfying $A = SBS^{-1}$ then $A^2 = SB^2S^{-1}$.

**Solution:**
Multiply out—

18) T F The set $\mathcal{B} = \{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$ is a basis for the kernel of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

**Solution:**
The vectors are not linearly independent.

19) | T | F | There exist two $2 \times 2$ matrices $A, B$ of rank 1 for which $AB$ has rank 2.

**Solution:**
The rank can not get larger

20) | T | F | Every basis of $\mathbf{R}^7$ contains exactly 7 vectors in it.

**Solution:**
The number of basis vectors is called the dimension. We have seen that this number does not depend on the choice of the basis.

| | Total

| Problem 2) (10 points) No justifications are needed. |
|---|

a) (3 points) Each of the following row reduced matrices are augmented matrices coming from a system of linear equations. Decide in each case whether the corresponding system has one, zero or infinitely many solutions and put a check mark in the appropriate column.

| Matrix | 0 solution | 1 solution | infinitely many solutions |
|---|---|---|---|
| $\left[\begin{array}{ccccc\|c} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$ | | | |

| Matrix | 0 solution | 1 solution | infinitely many solutions |
|---|---|---|---|
| $\left[\begin{array}{ccccc\|c} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$ | | | |

| Matrix | 0 solution | 1 solution | infinitely many solutions |
|---|---|---|---|
| $\left[\begin{array}{ccccc\|c} 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ | | | |

b) (4 points) Mark the sets which are linear spaces

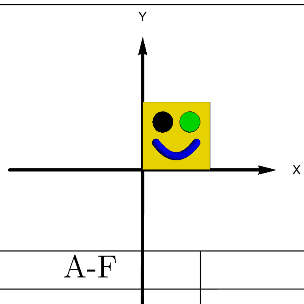
| | |
|---|---|
| | The set of polynomials $f$ in $P_2$ for which $f(2) = 3$. |
| | The set of functions $f$ in $C$ for which $f(2) = f(3)$. |
| | The set of vectors $\vec{x} = (x, y, z)$ in three dimensional space for which $x - y = z$ |
| | The kernel of the matrix $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. |

c) (3 points) Exactly one of the following matrices is a reflection, exactly one of the matrices is rotation by 90° and exactly one of them is a projection onto a subspace. Decide which is which.

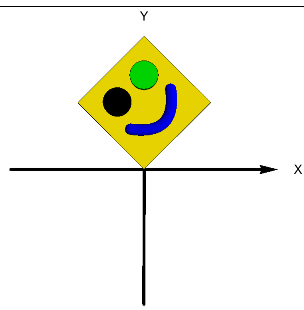
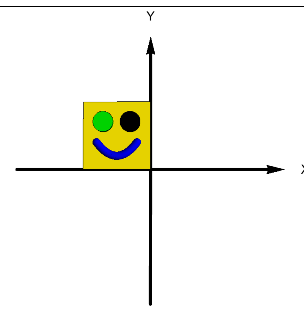
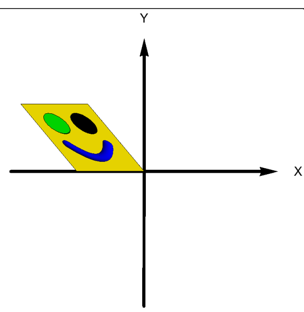
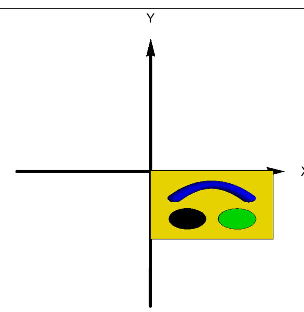
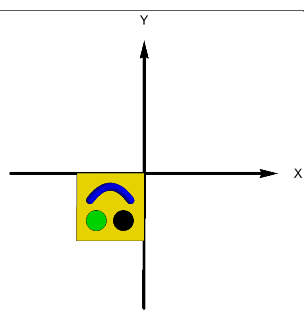
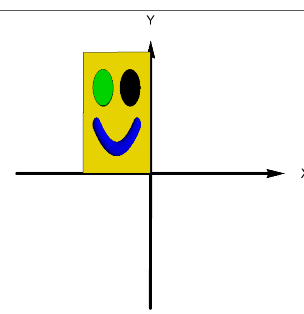
| Matrix | is a reflection | is a rotation by 90 | is a projection |
|---|---|---|---|
| $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ | | | |
| $\begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$ | | | |
| $\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ | | | |

**Solution:**
a) Infinitely.
Zero
Infinitely
b) All except the first are
c) reflection
projection
rotation

| Problem 3) (10 points) No justifications are necessary. |
|---|




a) (8 points) Each figure illustrate the action of a transformation $T(x) = Mx$, where $M$ is a $2 \times 2$ matrix. The picture to the left shows the case $M = I_2$. Match them with the matrices $A) - F)$. There is a one to one match.

| A-F | | A-F | |
|---|---|---|---|
| |  | |  |
| |  | |  |
| |  | |  |

A= $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ B= $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ C= $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ D= $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ E= $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ F= $\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$

b) (2 points) One of the following formulas is always true if $A, B, C$ are arbitrary invertible $7 \times 7$ matrices and $AXB = C$. Which one?

| | |
|---|---|
| $X = CA^{-1}B^{-1}$ | |
| $X = CB^{-1}A^{-1}$ | |
| $X = A^{-1}CB^{-1}$ | |
| $X = A^{-1}B^{-1}C$ | |
| $X = B^{-1}A^{-1}C$ | |

**Solution:**
a) First row : B C
second row: F E
third row: D A
b) The middle solution.
‘

Problem 4) (10 points)

Let's look at the system of equations

$$\begin{aligned} x + y + z + w &= 10 \\ x - y + z - w &= 2 \\ x + y - z - w &= 0 \end{aligned}$$

a) (3 points) Write the system in matrix form $A\vec{x} = \vec{b}$.

b) (7 points) Find all the solutions of the system.

**Solution:**

a) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 10 \\ 2 \\ 0 \end{bmatrix}$.

b) Row reduce the augmented matrix

$$B = [A|b] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10 \\ 1 & -1 & 1 & -1 & 2 \\ 1 & 1 & -1 & -1 & 0 \end{array}\right] .$$

The row reduced form is

$$B = [A|b] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 1 & 5 \end{array}\right] .$$

We have three leading 1 and one free variable. The solution is

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} .$$

Problem 5) (10 points)

Find a basis for the linear space $V$ of all vectors in $R^5$ perpendicular to $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. As usual, we want to see details of the computation.

**Solution:**
We have to find the kernel of a matrix which contains these two vectors in the rows.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \end{bmatrix} .$$

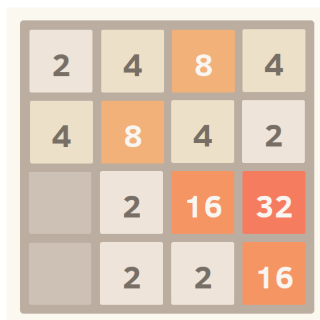
Row reducing gives

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} .$$

There are three free variables. A basis for the kernel is

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} .$$

Problem 6) (10 points)




The **2048 game** deals with $4 \times 4$ matrices. A couple of years ago, it became a viral hit. Oliver never succeeded in reaching 2048 and instead invented the "anti 2048 game", where the goal is to end the game with the least possible score. In the following matrix $A$, the coefficients of the matrix sum up to 96 points:

$$A = \begin{bmatrix} 2 & 4 & 8 & 4 \\ 4 & 8 & 4 & 2 \\ 2 & 4 & 8 & 4 \\ 4 & 8 & 4 & 2 \end{bmatrix}$$

a) (5 points) Find a basis for the kernel of $A$. (As usual, even if you should see the answer, we want you to derive the answer).

b) (5 points) Find a basis for the image of $A$. (Also here, we want you to tell how you got the answer).

**Solution:**
a) Row reduce, introduce free variables and collect. The basis for the kernel is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 2 \end{bmatrix} \right\} .$$

b) For the image, we just have to take the pivot columns of the matrix $A$, the columns which led to leading 1:

$$\left\{ \begin{bmatrix} 2 \\ 4 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 8 \\ 4 \\ 8 \\ 4 \end{bmatrix} \right\} .$$

| Problem 7) (10 points) |
| --- |

a) (3 points) Find an orthonormal basis $\mathcal{B}$ of $\mathbf{R}^3$ , which contains the two vectors

$$\vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} .$$

b) (7 points) Find the $3 \times 3$ matrix belonging to the transformation $T$ which reflects first about the line spanned by the vector $\vec{v}_1$ and then reflects about the line containing the vector $\vec{v}_2$.

**Solution:**

a) A third vector is $\begin{bmatrix} 1/\sqrt{2}, -1/\sqrt{2}, 0 \end{bmatrix}$. A basis is

$$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \right\} .$$

b) Write down the matrix $S$, the matrix $B$ and then form $A = SBS^{-1}$. In that basis the transformation is

$$B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And

$$A = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} .$$

| Problem 8) (10 points) |
|---|

**Computer vision** and **relativity** provide reasons why transformations in 4 dimensions can be important in applications. We look here at computer vision, where we can implement more general transformations by increasing the dimension.

a) (3 points) Decide whether the transformation

$$T(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

is a linear transformation or not. In any case, give a reason.

b) (3 points) Decide whether the transformation

$$S(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}) = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 1 & 0 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

is a linear transformation or not. In any case, give a reason.

c) (4 points) Verify that the first three coordinates of $S(\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix})$ agree with the three coordinates of $T(\begin{bmatrix} x \\ y \\ z \end{bmatrix})$.

**Solution:**

a) Since $T(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}) \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, the transformation is not linear.

b) We either check the three conditions, $T(0) = 0, T(v+w) = T(v)+T(w), T(\lambda v) = \lambda T(v)$ or just note that any transformation given by $T(x) = Ax$ is automatically linear.

c) Just compute the two vectors. The first is

$$S(\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}) = \begin{bmatrix} x + 2y + 3 \\ 2x + y + 2 \\ 2z + 4 \\ 1 \end{bmatrix} .$$

The second is

$$T(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x + 2y + 3 \\ 2x + y + 2 \\ 2z + 4 \end{bmatrix} .$$

| Problem 9) (10 points) |
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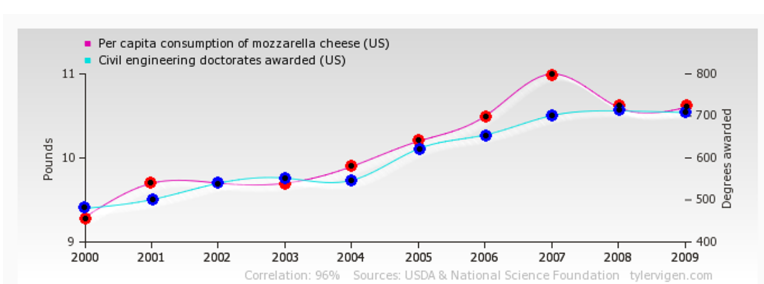
That **correlation does not necessarily imply causation** is one of the most important insights, when analyzing data. One has for example found a strong correlation between per capita consumption of **mozzarella cheese** $x$ and the number of **civil engineering doctorates** $y$ awarded in the US. Assume the data are given by the vectors $x = \begin{bmatrix} 9 \\ 10 \\ 14 \end{bmatrix}$ and $y = \begin{bmatrix} 500 \\ 600 \\ 700 \end{bmatrix}$. In the real data, the correlation has been computed to be 0.96 with 10 data points. Here we only take three data points.



a) (6 points) Find the vectors with coordinates $X_i = x_i - E[x], Y_i = y_i - E[y]$, where $E[x] = (x_1 + x_2 + x_3)/3$ is the **expectation**. The vectors $X, Y$ have now zero expectation. Also compute the dot product $X \cdot Y$, as well as the lengths $|X|, |Y|$. This corresponds up to a normalization to the **covariance** $\text{Cov}[X, Y] = X \cdot Y/3$ and **standard deviations** $\sigma[X] = |X|/\sqrt{3}, \sigma[Y] = |Y|/\sqrt{3}$ of the random variables $X, Y$ in statistics.

b) (4 points) Find the **correlation coefficient**

$$\frac{X \cdot Y}{|X||Y|} = \frac{\text{Cov}[X, Y]}{\sigma[X]\sigma[Y]}$$

which we know to be the cosine of the angle between $X$ and $Y$.

**Solution:**
a) The centered random vectors are

$$X = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}, Y = \begin{bmatrix} -100 \\ 0 \\ 100 \end{bmatrix}.$$

We have $X \cdot Y = 500$ and $|X| = \sqrt{14}$ and $|Y| = \sqrt{2}100$.
b) The correlation coefficient is $5/\sqrt{28}$.

| Problem 10) (10 points) |
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Let $V$ be the linear space spanned by the vectors

$$\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \}.$$

We have learned how to find a basis for the orthogonal complement $W = V^\perp$, the set of vectors perpendicular to $V$. While you should be able to do that we don't do that now. We ask you to find a basis for the orthogonal complement of $W$.

**Solution:**
Since the orthogonal complement of $W$ is $V$, we have to find a basis of $V$. If we had to find a basis of $S$ we had stuck the vectors as rows and row reduced $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ But we don't find here a basis for the kernel of that matrix. That would give us a basis for the orthogonal complement $W$. In order to find a basis for $V$ we want to find a basis of the vectors spanned by the three vectors. This means to find a basis for the image of $\begin{bmatrix} 1 & 4 & 1 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix}$. The row reduced form is $\begin{bmatrix} 1 & 0 & \frac{1}{5} \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We see that the first two columns were pivot columns and

$$\{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}\}$$

is a basis. P.S. If we would have asked you to find a basis for $W$, we would find a basis for the kernel of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

as you have done in a couple of homeworks. Then we would have row reduced the