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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) | T | F | Any 3×4 matrix A has exactly 3 columns.

Solution:
It has 4 columns.

2) | T | F | The set $\mathcal{B} = \{\begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix}\}$ is a basis for the linear space of 1×2-matrices.

Solution:
Yes, every matrix can be written as a combination of those

3) | T | F | If A is a 6×4 matrix and $Ax = 0$ has a unique solution, then the columns of A are linearly independent.

Solution:
The kernel is trivial if and only if columns are linearly independent

4) | T | F | If A is a 6×4 matrix and $Ax = 0$ has no nonzero solutions, then $Ax = e_1$ has a unique solution.

Solution:
(False: this has a solution only if e_1 is in the image of A

5) | T | F | There is a 2×4 matrix A such that $Ax = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ has a unique solution.

Solution:
(False: even if $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is in the image of A, the solution x will not be unique because A has nontrivial kernel.

6) | T | F | If A and B are invertible matrices mapping $\mathbb{R}^3 \to \mathbb{R}^3$, then $A + B$ is invertible.

Solution:
Take $A = I, B = -I$.

7) | T | F | The inverse of a product AB of invertible square matrices A, B is $A^{-1}B^{-1}$.

Solution:
Change the order

8) | T | F | The inverse of a horizontal shear is a horizontal shear.

Solution:
The shear constant in the off diagonal part just becomes its negative.

9) | T | F | If A, B are 2×2 matrices and $AB = 0$, then $BA = 0$.

Solution:
As a counter example, take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Now, $AB = 0$ but $BA = A$ is not zero.

10) | T | F | Given two 2×2 matrices, then the kernel of AB contains the kernel of A.

Solution:
AB contains the kernel of B but not necessarily the kernel of A. Take A to be the projection on the x axes and B the reflection at the line $x = y$. Then AB has e_1 in the kernel but not e_2, which was in the kernel of A.

11) | T | F | Given two 2×2 matrices, then the rank of AB is at most the rank of A.

Solution:
The image o AB is a subset of the image of A so that the rank has to be smaller or equal.

12) T F

There is a 2×2 matrix A such that $A \neq I_2$ and $A^3 = I_2$.

Solution:
Take a rotation about the origin by an angle 120 degrees.

13) T F

There is 2×2 matrix A not identical to 0 for which $A^2 = 0$.

Solution:
Take the matrix which has 1 in the top row and -1 in the second row.

14) T F

There is a 3×3 matrix B such that $B^2 \neq 0$ but $B^3 = 0$.

Solution:
Take the matrix which maps e_1 to 0 and e_2 to e_1 and e_3 to e_1. When doing it twice it maps e_1 to 0 and e_2 to 0 and e_3 to e_1. But applying B again kills everything and the matrix is a zero matrix.

15) T F

If $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$, then $A^8 = I$.

Solution:
The matrix A is not invertible. It is actually a projection onto $x = y$ and $A^2 = A$. So also $A^8 = A$.

16) T F

Given $A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$, the linear space of 2×2 matrices B such that $AB = 0$ has dimension 2.

Solution:
Writing $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we get $a + 2c = 0, b + 2d = 0$ which has a two dimensional solution space (2 free variables)

17) T F

If a 5×5 matrix A is invertible, then its nullity must be 0.

Solution:
The nullity is the dimension of the kernel.

18) T F — There is a 5×7 matrix A for which the rank of A is equal to the nullity of A.

Solution:
The sum of the rank and nullity is 7

19) T F — If e_1 is the first standard basis vector in $\mathbb{R}^4$, then the set of solutions of $Ax = e_1$ is a linear space.

Solution:
The solution space does not include 0.

20) T F — The subset of functions f in the space of continuous functions $C(\mathbb{R})$ which have the property that $f(7) \geq 0$ is a linear space.

Solution:
Check the three properties.

	Total

Problem 2) (10 points) No justifications are needed.

In all sub problems a-d), each mismatch is one point off until all points are depleted.

a) (3 points) Decide in each case whether the matrix is row reduced.

Matrix	is row reduced	is not row reduced
$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$		
$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$		
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$		
$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		

Solution:
The first three matrices are row reduced, the last one not.

b) (2 points) Mark the linear spaces

Check if linear space	Space
	The set of polynomials f in P_2 for which $\int_0^1 f(x)\,dx = 1$.
	The set of functions f in C^∞ for which $f'(0) \leq 0$.
	The set of functions f in C^∞ for which $f'(1) = 0$.
	The set of 3×3 matrices for which the product of all matrix entries is zero.

Solution:
Only the third one is a linear space. The first does not contain the zero function, the second can not be scaled by a negative number, the third fails addition. Take two matrices which have one zero at different places and 1 eveywhere else, then add them. This gives a matrix which has no zeros.

c) (3 points) Decide in each case whether the transformation satisfies the involution property $A^2 = I = I_3$, then whether it is invertible and then check whether it is a reflection at a 2-dimensional plane.

Matrix A	$A^2 = I$	A is invertible	A is a reflection at a plane
$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$			
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$			
$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$			
$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$			

Solution:
Lets look at the columns The first and last matrix satisfy $A^2 = I$. For the second column, all except the third are invertible. For the last column, only the first is a reflection at a plane. The last one is a reflection at a line because two directions get flipped.

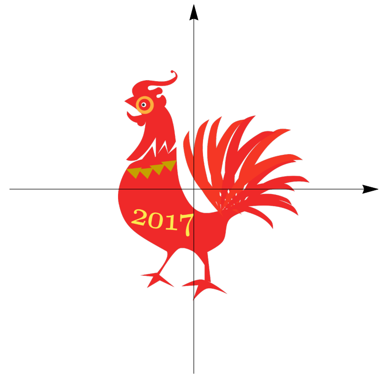
d) (2 points) Match the transformation type: (rotation dilation, reflection dilation, shear dilation, projection dilation). A **shear dilation** is a composition of a shear and a dilation. A **projection dilation** is a composition of an orthogonal projection and a dilation.

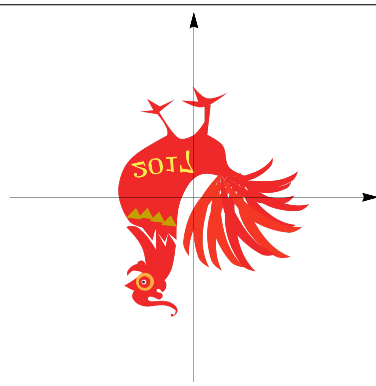
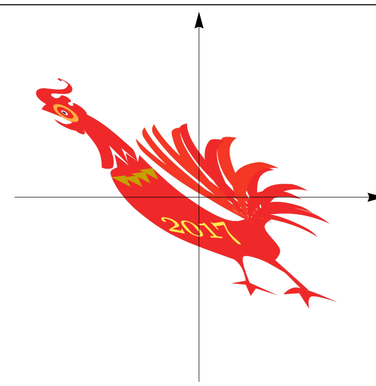
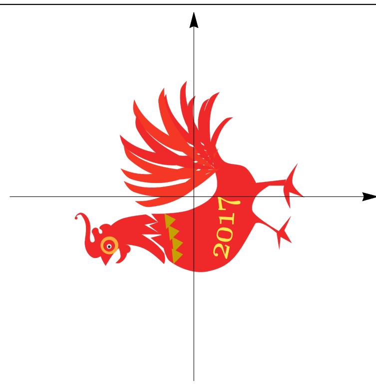
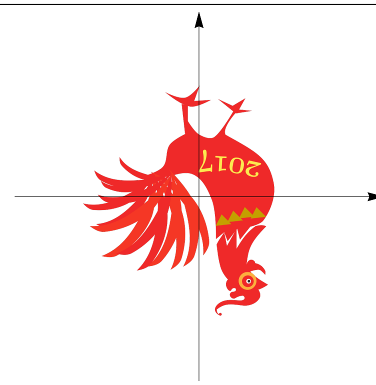
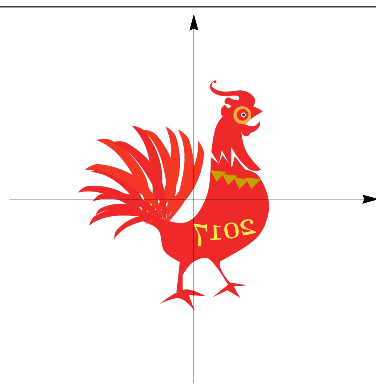
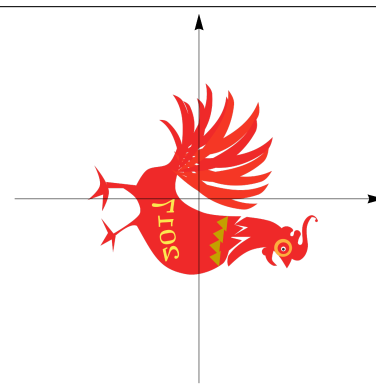
Fill in the name:				
Matrix	$\begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$	$\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$

Solution:
Projection-dilation, Shear-dilation, rotation-dilation,reflection-dilation

Problem 3) (10 points) No justifications are necessary.

The time between January 28, 2017 and February 15, 2018 is the **year of the rooster**. Match the effect of the transformation of the rooster with the matrix performing the transformation in the standard basis.
For the grading: each mismatch takes 2 points off until all 10 points are depleted.



A-F		A-F	
			
			
			

A=$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ B=$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ C=$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ D=$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ E=$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ F=$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

Solution:
E,F
A,D
C,B

Problem 4) (10 points)

a) (3 points) Write the system of equations

$$\begin{aligned} x + y + z &= 1 \\ x + y &= 0 \\ x + z &= 0 \end{aligned}$$

in the form $Av = b$, where A is a matrix and b is a vector and $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

b) (4 points) Find all the solutions to the system.

c) (3 points) Find the first column of the inverse matrix A^{-1}.

Solution:
a) Write it as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} .$$

b) Row reduce the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

to get the matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] .$$

and get the solution $[-1, 1, 1]^T$ which means $x = -1, y = 1, z = 1$.
c) Since $Ax = e_1$, we have $x = A^{-1}e_1$ meaning that x is the first column of the inverse matrix. Again we see that this is just $[-1, 1, 1]^T$.

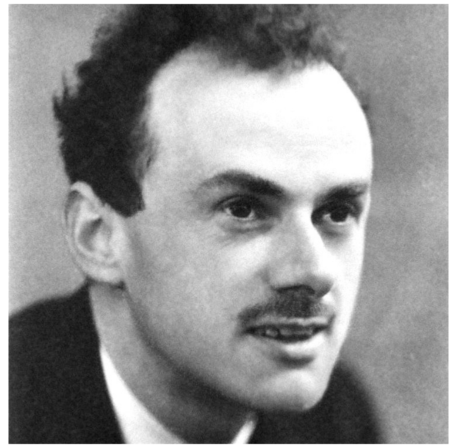
Problem 5) (10 points)

The matrices

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} C = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} D = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

are called the **Gamma matrices** or **Dirac matrices**. Here i is the imaginary $\sqrt{-1}$ which will be covered later in the course. Since we don't touch the matrix C here, you can ignore i for now.

The Dirac matrices have some nice properties. Lets discover some of them:



a) (3 points) Compute $AB + BA$.

b) (3 points) Compute $AD + DA$.

c) (2 points) Compute A^2.

d) (2 points) Compute D^2.

Solution:
These were just routine computations. But they show something interesting about the anticommutation of the matrices. In physics there are written as γ_j and the commutator is written as $\{A, B\} = AB + BA$. What actually happens is that $\{\gamma_j, \gamma_k\} = 2g_{jk}$, where $g = diag(1, -1, -1, -1)$ is the Lorentz metric. a) $AB + BA = 0$ the zero matrix
b) $AD + DA = 0$, the zero matrix
c) $A^2 = 1$, the identity matrix
d) $D^2 = -1$, minus the identity matrix

Problem 6) (10 points)

In the "checkers matrix", the entry 1 means that the checkers initial condition has a checker piece there and 0 means that that field is empty:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} .$$

a) (6 points) Find a basis for the kernel of A.

b) (4 points) Find a basis for the image of A.

Solution:
a) The kernel is 6 dimensional. To find it, row reduce the matrix.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

Introduce free variables a, b, c, d, e, f and write down the equtions $x = -a - c, y = -b - d, z = a, y = b, v = c, w = d, p = e, q = f$ to get

$$\begin{bmatrix} x \\ y \\ z \\ u \\ v \\ w \\ p \\ q \end{bmatrix} = a \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + e \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + f \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} ,$$

so that

$$\mathcal{B}_{ker}(A) = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} .$$

b) the image 2 dimensional. It is spanned by the first two vectors in the checkers matrix, because thats where the leading 1 were in $rref(A)$.

$$\mathcal{B}_{im}(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\} .$$

Problem 7) (10 points)

a) (5 points) Find a basis of the linear space W consisting of all vectors perpendicular to the two vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

b) (5 points) Find a basis of the space V of vectors perpendicular to W.

Solution:
a) Write the vectors as rows and find the kernel.
b) We know that the perpendicular space of the perpendicular space is the space itself. Therefore, the space V is spanned by the two original vectors. Either row reduce the matrix having them as columns or see that they are not parallel.

Problem 8) (10 points)

We have seen the following basis $\mathcal{B} = \{v_1, v_2, v_3, v_4\}$ in $\mathbb{R}^4$ already in the homework:

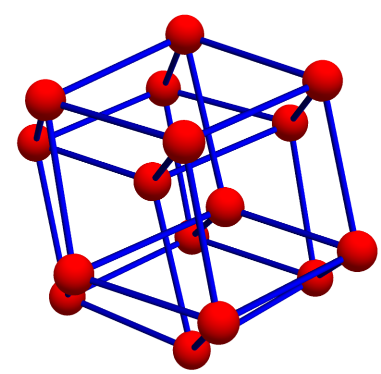
$$\mathcal{B} = \left\{ \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \right\}.$$

Lets look at the transformation T which maps $v_1 \to v_1$ and $v_2 \to v_3$ and $v_3 \to v_4$ and $v_4 \to v_2$. Let S be the matrix which contains the basis $\mathcal{B}$ as column vectors.

a) (3 points) Compute S^2 and use this to find the inverse of S.

b) (3 points) Find the matrix B of the transformation T given in the basis $\mathcal{B}$.

c) (4 points) Find the matrix A describing the transformation T in the standard basis e_1, e_2, e_3, e_4.



The transformation is a rotation in four dimensional space. We can use it to visualize the rotation of a four dimensional cube, the **tesseract**. The tesseract a popular object in mathematical pop culture.

Solution:
a) The matrix S is

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} /2 \,.$$

Since $S^2 = 1$, we have $S^{-1} = S$.
b) The matrix B is

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \,.$$

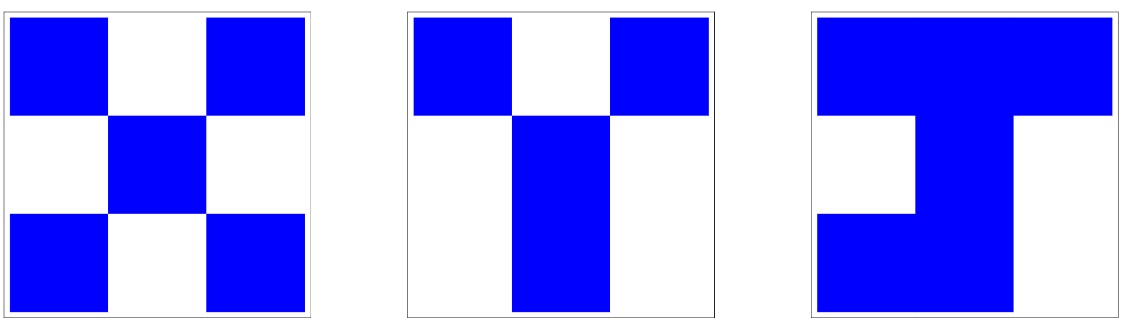
c) Now compute $A = SBS^{-1}$. Which gives

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \,.$$

This is the matrix of the transformation in the standard coordinates.

Problem 9) (10 points)

When doing optical character recognition (OCR) or face recognition, one looks at matrices encoding the letter or face and correlates them with known letters or faces. One then choses the one for which the correlation is the best. We built a small OCR system storing a letter as a vector in $\mathbb{R}^{10}$, where the last digit encodes the negative total area of the letter, so that the expectation is zero. Here are three symbols X, Y, Z,



where Z is a mystery character which needs identification:

$$X = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ -5 \end{bmatrix}, Y = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ -4 \end{bmatrix}, Z = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ -6 \end{bmatrix}.$$

In order to fit Z, we compute some correlations:

a) (1 point) Find $\text{Var}[X] = (X \cdot X)/10$ and standard deviation $\sigma[X] = \sqrt{\text{Var}[X]}$.

b) (1 point) Find $\text{Var}[Y] = (Y \cdot Y)/10$ and standard deviation $\sigma[Y] = \sqrt{\text{Var}[Y]}$.

c) (1 point) Find $\text{Var}[Z] = (Z \cdot Z)/10$ and standard deviation $\sigma[Z] = \sqrt{\text{Var}[Z]}$.

d) (2 points) Find the covariance $\text{Cov}[X, Z] = X \cdot Z/10$ of X and Z.

e) (2 points) Find the covariance $\text{Cov}[Y, Z] = Y \cdot Z/10$ of Y and Z.

f) (3 points) Compute $\text{Cor}[X, Z] = \frac{\text{Cov}[X,Z]}{\sigma[X]\sigma[Z]}$ and compute $\text{Cor}[Y, Z]$ in the same way. Which of the two is larger?

Solution:
a) 3, $\sqrt{3}$
b) 2, $\sqrt{2}$
c) $42/10 = 4.2$, $\sqrt{4.2}$
d) 3.4.
e) 2.8.
f) $3.4/(\sqrt{3}\sqrt{4.2})$ and $2.8/(\sqrt{2}\sqrt{4.2}$. The second one has a higher correlation.