Name:

MWF 9 Oliver Knill
MWF 10 Akhil Mathew
MWF 10 Ian Shipman
MWF 11 Rosalie Belanger-Rioux
MWF 11 Stephen Hermes
MWF 11 Can Kozcaz
MWF 11 Zhengwei Liu
MWF 12 Stephen Hermes
MWF 12 Hunter Spink
TTH 10 Will Boney
TTH 10 Changho Han
TTH 11:30 Brendan McLellan
TTH 11:30 Krishanu Sankar

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1)	T	F	Any 3×4 matrix A has exactly 3 columns.
2)	T	F	The set $\mathcal{B} = \{\begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix}\}$ is a basis for the linear space of 1×2-matrices.
3)	T	F	If A is a 6×4 matrix and $Ax = 0$ has a unique solution, then the columns of A are linearly independent.
4)	T	F	If A is a 6×4 matrix and $Ax = 0$ has no nonzero solutions, then $Ax = e_1$ has a unique solution.
5)	T	F	There is a 2×4 matrix A such that $Ax = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ has a unique solution.
6)	T	F	If A and B are invertible matrices mapping $\mathbb{R}^3 \to \mathbb{R}^3$, then $A + B$ is invertible.
7)	T	F	The inverse of a product AB of invertible square matrices A, B is $A^{-1}B^{-1}$.
8)	T	F	The inverse of a horizontal shear is a horizontal shear.
9)	T	F	If A, B are 2×2 matrices and $AB = 0$, then $BA = 0$.
10)	T	F	Given two 2×2 matrices, then the kernel of AB contains the kernel of A.
11)	T	F	Given two 2×2 matrices, then the rank of AB is at most the rank of A.
12)	T	F	There is a 2×2 matrix A such that $A \neq I_2$ and $A^3 = I_2$.
13)	T	F	There is 2×2 matrix A not identical to 0 for which $A^2 = 0$.
14)	T	F	There is a 3×3 matrix B such that $B^2 \neq 0$ but $B^3 = 0$.
15)	T	F	If $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$, then $A^8 = I$.
16)	T	F	Given $A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$, the linear space of 2×2 matrices B such that $AB = 0$ has dimension 2.
17)	T	F	If a 5×5 matrix A is invertible, then its nullity must be 0.
18)	T	F	There is a 5×7 matrix A for which the rank of A is equal to the nullity of A.
19)	T	F	If e_1 is the first standard basis vector in $\mathbb{R}^4$, then the set of solutions of $Ax = e_1$ is a linear space.
20)	T	F	The subset of functions f in the space of continuous functions $C(\mathbb{R})$ which have the property that $f(7) \geq 0$ is a linear space.

	Total

Problem 2) (10 points) No justifications are needed.

In all sub problems a-d), each mismatch is one point off until all points are depleted.

a) (3 points) Decide in each case whether the matrix is row reduced.

Matrix	is row reduced	is not row reduced
$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$		
$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$		
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$		
$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		

b) (2 points) Mark the linear spaces

Check if linear space	Space
	The set of polynomials f in P_2 for which $\int_0^1 f(x)\,dx = 1$.
	The set of functions f in C^∞ for which $f'(0) \leq 0$.
	The set of functions f in C^∞ for which $f'(1) = 0$.
	The set of 3×3 matrices for which the product of all matrix entries is zero.

c) (3 points) Decide in each case whether the transformation satisfies the involution property $A^2 = I = I_3$, then whether it is invertible and then check whether it is a reflection at a 2-dimensional plane.

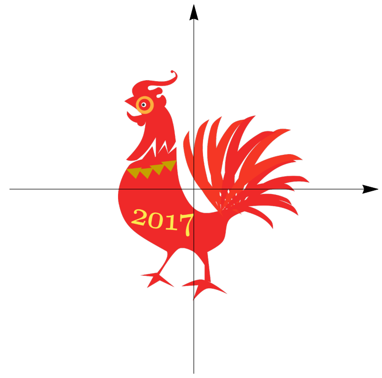
Matrix A	$A^2 = I$	A is invertible	A is a reflection at a plane
$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$			
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$			
$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$			
$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$			

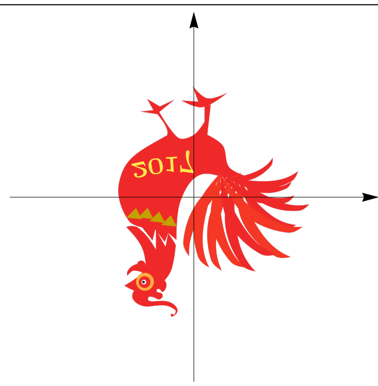
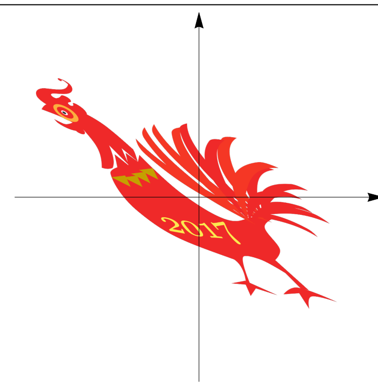
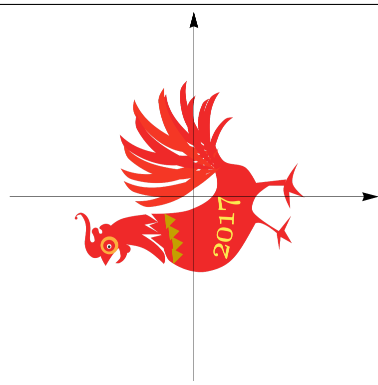
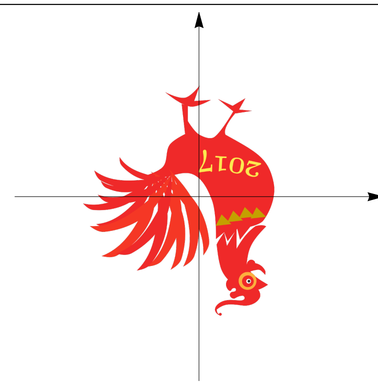
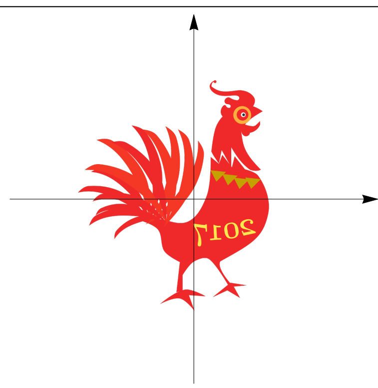
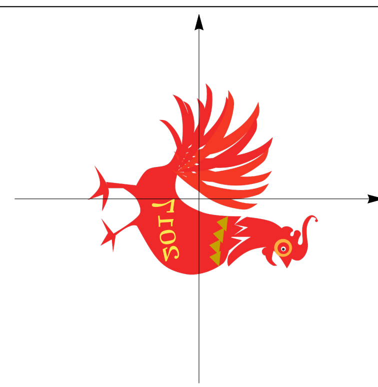
d) (2 points) Match the transformation type: (rotation dilation, reflection dilation, shear dilation, projection dilation). A **shear dilation** is a composition of a shear and a dilation. A **projection dilation** is a composition of an orthogonal projection and a dilation.

Fill in the name:				
Matrix	$\begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$	$\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$

Problem 3) (10 points) No justifications are necessary.

The time between January 28, 2017 and February 15, 2018 is the **year of the rooster**. Match the effect of the transformation of the rooster with the matrix performing the transformation in the standard basis.
For the grading: each mismatch takes 2 points off until all 10 points are depleted.



A-F		A-F	
			
			
			

$$A=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad B=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad C=\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad D=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad E=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad F=\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Problem 4) (10 points)

a) (3 points) Write the system of equations

$$\begin{array}{rcl} x+y+z & = & 1 \\ x+y & = & 0 \\ x+z & = & 0 \end{array}$$

in the form $Av = b$, where A is a matrix and b is a vector and $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

b) (4 points) Find all the solutions to the system.

c) (3 points) Find the first column of the inverse matrix A^{-1}.

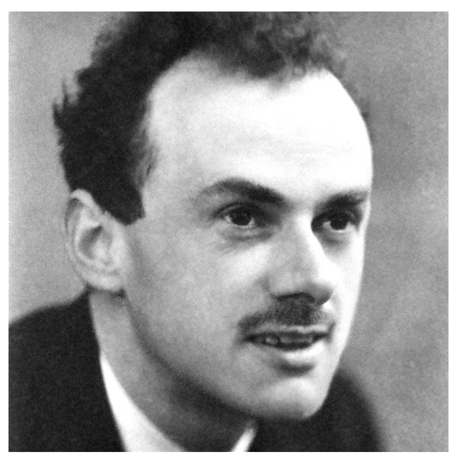
Problem 5) (10 points)

The matrices

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} C = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} D = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

are called the **Gamma matrices** or **Dirac matrices**. Here i is the imaginary $\sqrt{-1}$ which will be covered later in the course. Since we don't touch the matrix C here, you can ignore i for now.

The Dirac matrices have some nice properties. Lets discover some of them:



a) (3 points) Compute $AB + BA$.

b) (3 points) Compute $AD + DA$.

c) (2 points) Compute A^2.

d) (2 points) Compute D^2.

Problem 6) (10 points)

In the "checkers matrix", the entry 1 means that the checkers initial condition has a checker piece there and 0 means that that field is empty:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} .$$

a) (6 points) Find a basis for the kernel of A.

b) (4 points) Find a basis for the image of A.

Problem 7) (10 points)

a) (5 points) Find a basis of the linear space W consisting of all vectors perpendicular to the two vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} .$$

b) (5 points) Find a basis of the space V of vectors perpendicular to W.

Problem 8) (10 points)

We have seen the following basis $\mathcal{B} = \{v_1, v_2, v_3, v_4\}$ in $\mathbb{R}^4$ already in the homework:

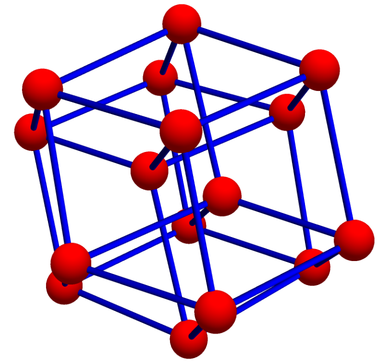
$$\mathcal{B} = \left\{ \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \right\} .$$

Lets look at the transformation T which maps $v_1 \to v_1$ and $v_2 \to v_3$ and $v_3 \to v_4$ and $v_4 \to v_2$. Let S be the matrix which contains the basis $\mathcal{B}$ as column vectors.

a) (3 points) Compute S^2 and use this to find the inverse of S.

b) (3 points) Find the matrix B of the transformation T given in the basis $\mathcal{B}$.

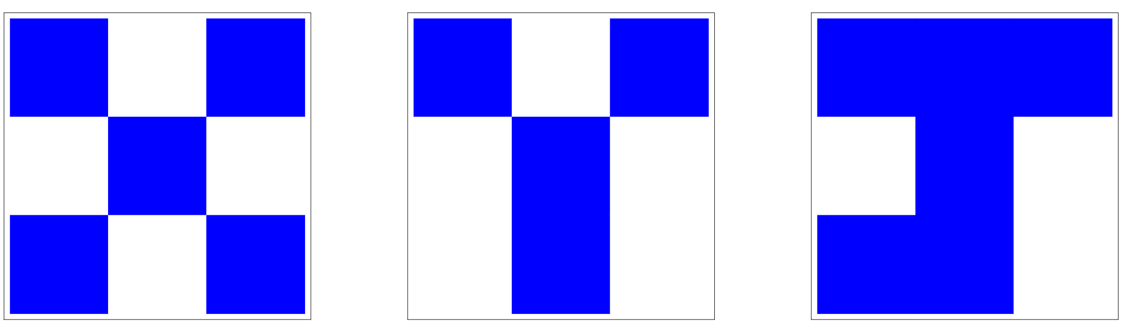
c) (4 points) Find the matrix A describing the transformation T in the standard basis e_1, e_2, e_3, e_4.



The transformation is a rotation in four dimensional space. We can use it to visualize the rotation of a four dimensional cube, the **tesseract**. The tesseract a popular object in mathematical pop culture.

Problem 9) (10 points)

When doing optical character recognition (OCR) or face recognition, one looks at matrices encoding the letter or face and correlates them with known letters or faces. One then choses the one for which the correlation is the best. We built a small OCR system storing a letter as a vector in $\mathbb{R}^{10}$, where the last digit encodes the negative total area of the letter, so that the expectation is zero. Here are three symbols X, Y, Z,



where Z is a mystery character which needs identification:

$$X = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ -5 \end{bmatrix}, Y = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ -4 \end{bmatrix}, Z = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ -6 \end{bmatrix}.$$

In order to fit Z, we compute some correlations:

a) (1 point) Find $\text{Var}[X] = (X \cdot X)/10$ and standard deviation $\sigma[X] = \sqrt{\text{Var}[X]}$.

b) (1 point) Find $\text{Var}[Y] = (Y \cdot Y)/10$ and standard deviation $\sigma[Y] = \sqrt{\text{Var}[Y]}$.

c) (1 point) Find $\text{Var}[Z] = (Z \cdot Z)/10$ and standard deviation $\sigma[Z] = \sqrt{\text{Var}[Z]}$.

d) (2 points) Find the covariance $\text{Cov}[X, Z] = X \cdot Z/10$ of X and Z.

e) (2 points) Find the covariance $\text{Cov}[Y, Z] = Y \cdot Z/10$ of Y and Z.

f) (3 points) Compute $\text{Cor}[X, Z] = \frac{\text{Cov}[X,Z]}{\sigma[X]\sigma[Z]}$ and compute $\text{Cor}[Y, Z]$ in the same way. Which of the two is larger?