# Homework 31: Partial differential equations

This is the last homework. It is due on the last day of classes Wednesday April 26, respectively on Tuesday, April 25, 2017.

1 Solve the heat equation $f_t = 17 f_{xx}$ on $[0, \pi]$ with the initial condition $f(x,0) = \max(\cos(x), 0)$.

2 Solve the partial differential equation $u_t = u_{xxxxxx} + u_{xx}$ with initial condition $u(0,x) = x^3$.

3 A piano string is fixed at the ends $x = 0$ and $x = \pi$ and is initially undisturbed $u(x,0) = 0$. The piano hammer induces an initial velocity $u_t(x,0) = g(x)$ onto the string, where $g(x) = \sin(3x)$ on the interval $[0, \pi/2]$ and $g(x) = 0$ on $[\pi/2, \pi]$. How does the string amplitude $u(x,t)$ move, if it follows the wave equation $u_{tt} = u_{xx}$?

4 A laundry line is excited by the wind. It satisfies the differential equation $u_{tt} = u_{xx} + \cos(t) + \cos(3t)$. Assume that the amplitude $u$ satisfies initial condition $u(x,0) = 4\sin(5x) + 10\sin(6x)$ and that its initial velocity is zero. Find the function $u(x,t)$ which satisfies the differential equation.
Hint. First find the general homogeneous solution $u_{homogeneous}$ of $u_{tt} = u_{xx}$ for an odd $u$ then a particular solution $u_{particular}(t)$ which only depends on $t$. Finally fix the Fourier coefficients.

5 In this course we have looked at four different types of differential equations. Systems of linear differential equations $x' = Ax$, nonlinear equations $x' = f(x,y), y' = g(x,y)$, inhomogeneous equations $p(D)f = g$ as well as partial differential equations like $u_t = D^2 u$ and wave equation $u_{tt} = D^2 u$. Give a concrete example of each type, in each case an example which has never appeared in any homework nor handout.

## Partial differential equations

In all these PDE problems, we look at functions $f$ on the interval $[0, \pi]$ and write them as a sin-series which means that we only need to compute the $b_n$ using the formula

$$\frac{2}{\pi} \int_0^\pi f(x) \sin(nx) \, dx \ .$$

This is justified as we can think of $f$ continued as $f(-x) = -f(x)$ on $[-\pi, 0]$ The temperature distribution $f(x,t)$ in a metal bar $[0, \pi]$ satisfies the **heat equation**

$$f_t(x,t) = \mu f_{xx}(x,t) = D^2 f(x,t)$$

Here $\mu$ is a positive constant which depends on the material. The height of a string $f(x,t)$ at time $t$ and position $x$ on $[0, \pi]$ satisfies the **wave equation**

$$f_{tt}(t,x) = c^2 f_{xx}(t,x) = c^2 D^2(f,t)$$

Here $c$ is a positive constant which tell how fast the waves move. All these problems are solved by diagonalizing $D^2$ using a Fourier basis. In the heat equation, write the initial condition as a Fourier series and write down the solution. In the wave equation, write both the initial condition $f(0,x)$ as well as the initial velocity $f_t(0,x)$ as a Fourier series and write down the solutions.