Homework 13: Gram Schmidt and QR

This homework is due on Friday, March 3, respectively on Tuesday, March 7, 2017.

1 Perform the Gram-Schmidt process on the two vectors $\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}$.

2 Find an orthonormal basis of the hyper plane $x_1+x_2+x_3+x_4 = 0$ in $\mathbf{R}^4$.

3 As in the previous problem, we first find an orthonormal basis of the kernel of

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 \end{bmatrix}.$$

Then find an orthonormal basis for the image.

4 Find the QR factorization of the following three matrices

$$A = \begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & 0 \\ 7 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, C = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}.$$

5 Find the QR factorization of the following matrices:

$$A = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 6 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Gram Schmidt and QR

The **Gram Schmidt orthogonalization process** produces from an arbitrary basis $\mathcal{B} = \{v_j\}$ an orthonormal basis $\{u_j\}$. This goes as follows: $\vec{w}_1 = \vec{v}_1$ and $\vec{u}_1 = \vec{w}_1/|\vec{w}_1|$. To construct u_i once you've already constructed $\vec{u}_1, \ldots, \vec{u}_{i-1}$ so that they are orthonormal, make the new vector $\vec{w}_i = \vec{v}_i - \text{proj}_{V_{i-1}}(\vec{v}_i)$, where $V_{i-1} = \text{span}(u_1, \ldots, u_{i-1})$, and then normalize $\vec{w}_i$ to to get $\vec{u}_i = \vec{w}_i/|\vec{w}_i|$. Then $\{\vec{u}_1, \ldots, \vec{u}_n\}$ is an orthonormal basis of V and the formulas

$$\vec{v}_1 = |\vec{v}_1|\vec{u}_1 = r_{11}\vec{u}_1$$

$$\vdots$$

$$\vec{v}_i = (\vec{u}_1 \cdot \vec{v}_i)\vec{u}_1 + . + (\vec{u}_{i-1} \cdot \vec{v}_i)\vec{u}_{i-1} + |\vec{w}_i|\vec{u}_i = r_{1i}\vec{u}_1 + .. + r_{ii}\vec{u}_i$$

$$\vdots$$

$$\vec{v}_n = (\vec{u}_1 \cdot \vec{v}_n)\vec{u}_1 + . + (\vec{u}_{n-1} \cdot \vec{v}_n)\vec{u}_{n-1} + |\vec{w}_n|\vec{u}_n = r_{1n}\vec{u}_1 + .. + r_{nn}\vec{u}_n$$

can be written in matrix form as $A = QR$,

$$\begin{bmatrix} | & | & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & | & | \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ 0 & 0 & \cdots & r_{nn} \end{bmatrix},$$

where A and Q are $m \times n$ matrices and R is an $n \times n$ matrix.