DOT PRODUCT. With the **dot product** in $\mathbf{R}^n$, we were able to define **angles**, **length**, compute projections onto planes or reflections on lines. Especially recall that if $\vec{w}_1, ..., \vec{w}_n$ was an orthonormal set, then $\vec{v} = a_1\vec{w}_1 + ... + a_n\vec{w}_n$ with $a_i = \vec{v} \cdot \vec{w}_i$. This was the formula for the orthonormal projection in the case of an orthogonal set. We will aim to do the same for functions. But first we need to define a "dot product" for functions.

THE INNER PRODUCT. For piecewise smooth functions f, g on $[-\pi, \pi]$, we define the **inner product**

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\, dx$$

It plays the role of the dot product in $\mathbf{R}^n$. It has the same properties as the familiar dot product:

(i) $\langle f+g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$.
(ii) $||f||^2 = \langle f, g\rangle \geq 0$
(iii) $||f||^2 = 0$ if and only if f is identically 0

EXAMPLES.

- $f(x) = x^2$ and $g(x) = \sqrt{x}$. Then $\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} x^{3/2}\, dx = \frac{1}{\pi}x^{5/2}\frac{2}{5}|_{-\pi}^{\pi} = \frac{4}{5}\sqrt{\pi^3}$.
- $f(x) = \sin^2(x)$, $g(x) = x^3$. Then $\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2(x)x^3\, dx = ...?$

Before integrating, Ii is always a good idea to look for some symmetry. Can you see the result without doing the integral?

PROPERTIES. The

- **triangle inequality** $||f+g|| \leq ||f|| + ||g|$.
- the **Cauchy-Schwartz inequality** $|\langle f, g\rangle \leq ||f||\ ||g||$
- as well as **Pythagoras theorem** $||f+g||^2 = ||f||^2 + ||g||^2$ for orthogonal functions

hold in the same way as they did in R^n. The proofs are identical.

ANGLE, LENGTH, DISTANCE, ORTHOGONALITY. With an inner product, we can do things as with the dot product:

- Compute the **angle** α between two functions f and g $\boxed{\cos(\alpha) = \frac{\langle f,g\rangle}{||f||\cdot||g||}}$
- Determine the **length** $\boxed{||f||^2 = \langle f, f\rangle}$
- Find and **distance** $\boxed{||f-g||}$ between two functions
- Project a function f onto a space of functions. $P(f) =< f, g_1 > g_1 + < f, g_2) g_2 ... + < f, g_n > g_n$ if the functions g_i are orthonormal.

Note that $||f|| = 0$ implies that f is identically 0. Two functions whose distance is zero are identical.

EXAMPLE: ANGLE COMPUTATION.

Problem: Find the angle between the functions $f(t) = t^3$ and $g(t) = t^4$.

Answer: The angle is $90°$. This can be seen by symmetry. The integral on $[-\pi, 0]$ is the negative then the integral on $[0, \pi]$.

EXAMPLE: GRAM SCHMIDT ORTHOGONALIZATION.

Problem: Given a two dimensional plane spanned by $f_1(t) = 1, f_2(t) = t^2$, use Gram-Schmidt orthonormalization to get an orthonormal set.

Solution. The function $g_1(t) = 1/\sqrt{2}$ has length 1. To get an orthonormal function $g_2(t)$, we use the formula of the Gram-Schmidt orthogonalization process: first form

$$h_2(t) = f_2(t) - \langle f_2(t), g_1(t)\rangle g_1(t)$$

then get $g_2(t) = h_2(t)/||h_2(t)||$.

EXAMPLE: PROJECTION.

Problem: Project the function $f(t) = t$ onto the plane spanned by the functions $\sin(t), \cos(t)$.

EXAMPLE: REFLECTION.

Problem: Reflect the function $f(t) = \cos(t)$ at the line spanned by the function $g(t) = t$.
Solution: Let $c = ||g||$. The projection of f onto g is $h = \langle f, g\rangle g/c^2$. The reflection is $f + 2(h - f)$ as with vectors.

EXAMPLE: Verify that if $f(t)$ is a 2π periodic function, then f and its derivative f' are orthogonal.

Solution. Define $g(x, t) = f(x + t)$ and consider its length $l(t) = ||g(x, t)||$ when fixing t. The length does not change. So, differentiating $0 = l'(t) = d/dt\langle f(x + t), f(x + t)\rangle = \langle f'(x + t), f(x + t)\rangle + \langle f(x + t), f'(x + t)\rangle = 2\langle f'(x + t), f(x + t)\rangle$.

PROBLEMS.

1. Find the angle between $f(x) = \cos(x)$ and $g(x) = x^2$. (Like in $\mathbb{R}^n$, we define the angle between f and g to be $\arccos \frac{\langle f,g\rangle}{\|f\|\|g\|}$ where $\|f\| = \sqrt{\langle f, f\rangle}$.)

   Remarks. Use integration by parts twice to compute the integral. This is a good exercise if you feel a bit rusty about integration techniques. Feel free to double check your computation with the computer but try to do the computation by hand.

2. A function on $[-\pi, \pi]$ is called **even** if $f(-x) = f(x)$ for all x and **odd** if $f(-x) = -f(x)$ for all x. For example, $f(x) = \cos x$ is even and $f(x) = \sin x$ is odd.

   a) Verify that if f, g are even functions on $[-\pi, \pi]$, their inner product can be computed by $\langle f, g\rangle = \frac{2}{\pi}\int_0^\pi f(x)g(x)\, dx$.

   b) Verify that if f, g are odd functions on $[-\pi, \pi]$, their inner product can be computed by $\langle f, g\rangle = \frac{2}{\pi}\int_0^\pi f(x)g(x)\, dx$.

   c) Verify that if f is an even function on $[-\pi, \pi]$ and g is an odd function on $[-\pi, \pi]$, then $\langle f, g\rangle = 0$.

3. Which of the two functions $f(x) = \cos(x)$ or $g(x) = \sin(x)$ is closer to the function $h(x) = x^2$?

4. Determine the projection of the function $f(x) = x^2$ onto the "plane" spanned by the two orthonormal functions $g(x) = \cos(x)$ and $h(x) = \sin(x)$.

   Hint. You have computed the inner product between f and g already in problem 1). Think before you compute the inner product between f and h. There is no calculation necessary to compute $\langle f, h\rangle$.

5. Recall that $\cos(x)$ and $\sin(x)$ are orthonormal. Find the length of $f(x) = a\cos(x) + b\sin(x)$ in terms of a and b.