

A linear space  $X$  has the property that if  $f, g$  are in  $X$ , then  $f + g, \lambda f$  and a zero vector"  $0$  are in  $X$ .

- $P_n$ , the space of all polynomials of degree  $n$ .
- $C^\infty$ , the space of all smooth functions.
- The space  $P$  of all polynomials.
- $C_{per}^\infty$  the space of all  $2\pi$  periodic functions.

In all these function spaces, the function  $f(x) = 0$  which is constant  $0$  is the zero function.

A map  $T$  on a linear space  $X$  is called a **linear transformation** if the following three properties hold

$T(x + y) = T(x) + T(y), T(\lambda x) = \lambda T(x)$  and  $T(0) = 0$ . Examples are:

- $Df(x) = f'(x)$  on  $C^\infty$
- $Tf(x) = \sin(x)f(x)$  on  $C^\infty$
- $Tf(x) = \int_0^x f(x) dx$  on  $C^\infty$
- $Tf(x) = 5f(x)$
- $Tf(x) = f(2x)$
- $Tf(x) = f(x - 1)$
- $Tf(x) = \sin(x)f(x)$
- $Tf(x) = e^t \int_0^x e^{-t} f(t) dt$

Subspaces, eigenvalues, basis, kernel and image are defined as before

$X$ linear subspace	$f, g \in X, f + g \in X, \lambda f \in X, 0 \in X$ .
$T$ linear transformation	$T(f + g) = T(f) + T(g), T(\lambda f) = \lambda T(f), T(0) = 0$ .
$f_1, f_2, \dots, f_n$ linear independent	$\sum_i c_i f_i = 0$ implies $f_i = 0$ .
$f_1, f_2, \dots, f_n$ span $X$	Every $f$ is of the form $\sum_i c_i f_i$ .
$f_1, f_2, \dots, f_n$ basis of $X$	linear independent and span.
$T$ has eigenvalue $\lambda$	$Tf = \lambda f$
kernel of $T$	$\{Tf = 0\}$
image of $T$	$\{Tf \mid f \in X\}$ .

Some concepts do not work without modification. Example:  $\det(T)$  or  $\text{tr}(T)$  are not always defined for linear transformations in infinite dimensions. The concept of a basis in infinite dimensions also needs to be defined properly.

The linear map  $Df(x) = f'(x)$  can be iterated:  $D^n f = f^{(n)}$  is the  $n$ 'th derivative. It is a differential operator which allows to write differential equations like  $f'' - f' = g$  in the same way than systems  $A\vec{x} = \vec{b}$ .

The kernel of  $D$  on  $C^\infty$  consists of all functions which satisfy  $f'(x) = 0$ . These are the constant functions. The kernel is one dimensional. The image is the entire space  $X$  because we can solve  $Df = g$  by integration. You see that in infinite dimension, the fact that the image is equal to the codomain is not equivalent that the kernel is trivial.

Example: solve

$$Df = g .$$

The linear transformation  $T$  has a one dimensional kernel, the linear space of constant functions. The system  $Df = g$  has therefore infinitely many solutions. Indeed, the solutions are of the form  $f = G + c$ , where  $F$  is the anti-derivative of  $g$ .

To solve

$$T(f) = (D - \lambda)f = g .$$

one can find  $f$  with the important formula

$$f(x) = Ce^{\lambda x} + e^{\lambda x} \int_0^x g(t)e^{-\lambda t} dt$$

as one can see by differentiating  $f$ : check  $f' = \lambda f + g$ . This is an important step because if we can invert  $T$ , we can invert also products  $T_k T_{k-1} \dots T_1$  and so solve  $p(D)f = g$  for any polynomial  $p$ .

Find the eigenvectors to the eigenvalue  $\lambda$  of the operator  $D$  on  $C^\infty(T)$ . We have to solve

$$Df = \lambda f .$$

We see that  $f(x) = e^{\lambda x}$  is a solution. But it is only a periodic solution if  $\lambda = 2k\pi i$ . Every number  $\lambda = 2\pi k i$  is an eigenvalue. Eigenvalues are "quantized".

When we solved the harmonic oscillator differential equation

$$D^2 f + f = 0 .$$

last week, we actually saw that the transformation  $T = D^2 + 1$  has a two dimensional kernel. It is spanned by the functions  $f_1(x) = \cos(x)$  and  $f_2(x) = \sin(x)$ . Every solution to the differential equation is of the form  $c_1 \cos(x) + c_2 \sin(x)$ .

**Problem 1)** Which of the following maps are linear operators?

- a)  $T(f)(x) = x^2 f(x - 4)$
- b)  $T(f)(x) = f'(x)^2$
- c)  $T = D^2 + D + 1$  meaning  $T(f)(x) = f''(x) + f'(x) + f(x)$ .
- d)  $T(f)(x) = e^x \int_0^x e^{-t} f(t) dt$ .

**Problem 2)** a) What is the kernel and image of the linear operators  $T = D + 3$  and  $D - 2$ ? Use this to find the kernel of  $p(D)$  for  $p(x) = x^2 + x - 6$ ?  
b) Verify that  $f(x) = xe^{-x^2/2}$  is in the kernel of the differential operator  $T = D + x$ .

**Problem 3)** In quantum mechanics, the operator  $P = iD$  is called the **momentum operator** and the operator  $Qf(x) = xf(x)$  is the **position operator**.

- a) Verify that every  $\lambda$  is an eigenvalue of  $P$ . What is the eigenfunction?
- b) What operator is  $[Q, P] = QP - PQ$ ?

**Problem 4)** The differential equation  $f' - 3f = \sin(x)$  can be written as

$$Tf = g$$

with  $T = D - 3$  and  $g = \sin$ . We need to invert the operator  $T$ . Verify that

$$Hg = e^{3x} \int_0^x e^{-3t} g(t) dt$$

is an inverse of  $T$ . In other words, show that the function  $f = Hg$  satisfies  $Tf = g$ .

**Problem 5)** The operator

$$Tf(x) = -f''(x) + x^2 f(x)$$

is called the **energy operator** of the **quantum harmonic oscillator**.

- a) Check that  $f(x) = e^{-x^2/2}$  is an eigenfunction of  $T$ . What is the eigenvalue?
- b) Verify that  $f(x) = xe^{-x^2/2}$  is an eigenfunction of  $T$ . What is the eigenvalue?