7.4.16 Diagonalizable. The eigenvalues are 3,2,1, with associated eigenvectors $\begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$.

If we let $S = \begin{bmatrix}2 & 1 & 0\\0 & 0 & 1\\1 & 1 & 0\end{bmatrix}$, then $S^{-1}AS = D = \begin{bmatrix}3 & 0 & 0\\0 & 2 & 0\\0 & 0 & 1\end{bmatrix}$.

7.4.28 Diagonalizable for positive values of a. The characteristic polynomial is $-\lambda^3 + a\lambda = -\lambda(\lambda^2 - a)$. If a is positive, then we have three distinct real eigenvalues, $0, \pm\sqrt{a}$, so that the matrix will be diagonalizable. If a is negative or 0, then 0 is the only real eigenvalue, and the matrix fails to be diagonalizable.

7.4.36 Yes. The matrices $\begin{bmatrix}-1 & 6\\-2 & 6\end{bmatrix}$ and $\begin{bmatrix}1 & 2\\-1 & 4\end{bmatrix}$ both have the eigenvalues 3 and 2, so that each of them is similar to the diagonal matrix $\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}$, by Algorithm 7.4.4. Thus $\begin{bmatrix}-1 & 6\\-2 & 6\end{bmatrix}$ is similar to $\begin{bmatrix}1 & 2\\-1 & 4\end{bmatrix}$, by parts b and c of Theorem 3.4.6.

7.4.50 The matrix of T with respect to the standard basis $1, x, x^2$ is $B = \begin{bmatrix}1 & -3 & 9\\0 & 1 & -6\\0 & 0 & 1\end{bmatrix}$.

The only eigenvalue of B is 1, with corresponding eigenvector $\begin{bmatrix}1\\0\\0\end{bmatrix}$. The only eigenvalue of T is 1 as well, with corresponding eigenfunction $f(x) = 1$. T fails to be diagonalizable, since there is only one eigenvalue, with a one-dimensional eigenspace.

7.4.54 Note that $A^2 = 0$, but $B^2 \neq 0$. Since A^2 fails to be similar to B^2, matrix A isn't similar to B (see Example 7 of Section 3.4).

7.4.58N a If $\vec{v} \in \text{im}(A)$, then $\vec{v} = A\vec{u}$ for some vector $\vec{u}$. Then $A\vec{v} = A^2\vec{u} = \vec{0}$ showing that $\vec{0}$ is in the kernel of A.

b Because the matrix is nonzero, the image is at least one-dimensional. It can not be of larger dimension, because otherwise, the kernel would be 1 dimensional or less by the rank-nullity theorem and the image could not be a subspace as established in 7.4.58a. To summarize, the image is one dimensional and the kernel is 2 dimensional.

c By assumption, the vectors $\vec{v}_1, \vec{v}_3$ are linearly independent and form a basis of the kernel of A. To establish the claim, we only have to show that $\vec{v}_2$ is not in the kernel of A. If it were, then $\vec{v}_1$ would be $\vec{0}$ contradicting the assumption.

d $T\vec{v}_1 = \vec{0}, T\vec{v}_2 = \vec{v}_1, T\vec{v}_3 = \vec{0}$ shows

$$B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

7.4.56 The hint shows that matrix $M = \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}$ is similar to $N = \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$; thus matrices M and N have the same characteristic polynomial, by Theorem 7.3.6a. Now $f_M(\lambda) = \det \begin{bmatrix} AB - \lambda I_n & 0 \\ B & -\lambda I_n \end{bmatrix} = (-\lambda)^n \det(AB - \lambda I_n) = (-\lambda)^n f_{AB}(\lambda)$. To understnd the second equality, consider Theorem 6.1.5. Likewise, $f_N(\lambda) = (-\lambda)^n f_{BA}(\lambda)$. It follows that $(-\lambda)^n f_{AB}(\lambda) = (-\lambda)^n f_{BA}(\lambda)$ and therefore $f_{AB}(\lambda) = f_{BA}(\lambda)$, as claimed.

Ch 7.TF.**23** T; The sole eigenvalue, 7, must have geometric multiplicity 3.

Ch 7.TF.**29** F; Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, for example.