4.1.6 Not a subspace, since I_3 and $-I_3$ are invertible, but their sum is not.

4.1.7 The set V of diagonal 3×3 matrices is a subspace of $\mathbb{R}^{3\times 3}$:

4.1.7a The zero matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is in V,

4.1.7b If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ and $B = \begin{bmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{bmatrix}$ are in V, then so is their sum

$$A + B = \begin{bmatrix} a+p & 0 & 0 \\ 0 & b+q & 0 \\ 0 & 0 & c+r \end{bmatrix}.$$

4.1.7c If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ is in V, then so is $kA = \begin{bmatrix} ka & 0 & 0 \\ 0 & kb & 0 \\ 0 & 0 & kc \end{bmatrix}$, for all constants k.

4.1.8 This is a subspace; the justification is analogous to Exercise 7.

4.1.9 Not a subspace; consider multiplication with a negative scalar. I_3 belongs to the set, but $-I_3$ doesn't.

4.1.10 Let $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Let V be the set of all 3×3 matrices A such that $A\vec{v} = \vec{0}$. Then V is a subspace of $\mathbb{R}^{3\times 3}$:

4.1.10a The zero matrix 0 is in V, since $0\vec{v} = \vec{0}$.

4.1.10b If A and B are in V, then so is $A + B$, since $(A + B)\vec{v} = A\vec{v} + B\vec{v} = \vec{0} + \vec{0} = \vec{0}$.

4.1.10c If A is in V, then so is kA for all scalars k, since $(kA)\vec{v} = k(A\vec{v}) = k\vec{0} = \vec{0}$.

4.1.11 Not a subspace: I_3 is in rref, but the scalar multiple $2I_3$ isn't.

4.1.36 Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. We want $AB = BA$, or

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \text{ or}$$

$$\begin{bmatrix} 2a & 3b & 3c \\ 2d & 3e & 3f \\ 2g & 3h & 3i \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ 3d & 3e & 3f \\ 3g & 3h & 3i \end{bmatrix}.$$

We note that b, c, d, g must be zero, but a, e, f, h and i are chosen freely. So, our space, V, consists of all matrices of the form $\begin{bmatrix} a & 0 & 0 \\ 0 & e & f \\ 0 & h & i \end{bmatrix}$

$$= a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + f \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + h \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a basis of V, and $\dim(V) = 5$.

4.1.48 If $f(t) = a + bt + ct^2 + dt^3 + et^4$, then $f(-t) = a - bt + ct^2 - dt^3 + et^4$ and $-f(-t) = -a + bt - ct^2 + dt^3 - et^4$.

4.1.48 a f is even if $f(-t) = f(t)$ for all t. Comparing coefficients we find that $b = d = 0$, so that $f(t)$ is of the form $f(t) = a + ct^2 + et^4$, with basis $1, t^2, t^4$. The dimension is 3.

b f is odd if $f(-t) = -f(t)$, which is the case if $a = c = e = 0$. The odd polynomials are of the form $f(t) = bt + dt^3$, with basis t, t^3 and dimension 2.

4.1.58 a Let $g(x)$ be a function in V. Thus, $g''(x) = -g(x)$. Now, if $f(x) = g(x)^2 + g'(x)^2$, then $f'(x) = 2(g(x))g'(x) + 2(g'(x))g''(x) = 2(g(x))g'(x) - 2(g(x))g'(x) = 0$. So, $f(x) = g(x)^2 + g'(x)^2$ is a constant function.

b Let $g(x)$ be a function in V such that $g(0) = g'(0) = 0$. From part a we know that $g(x)^2 + g'(x)^2 = k$, a constant. Now $g(0)^2 + g'(0)^2 = 0^2 + 0^2 = 0$, so that $k = 0$. The equation $g(x)^2 + g'(x)^2 = 0$ means that $g(x) = g'(x) = 0$ for all x, as claimed.

c First we note that $g(x) = f(x) - f(0)\cos(x) - f'(0)\sin(x)$ is in V, since the functions $f(x), \cos(x)$ and $\sin(x)$ are all in V, and V is a subspace of $F(\mathbb{R}, \mathbb{R})$. Note that $g(0) = f(0) - f(0)\cos(0) - f'(0)\sin(0) = f(0) - f(0) = 0$. Also, $g'(x) = f'(x) + f(0)\sin(x) - f'(0)\cos(x)$, so that $g'(0) = f'(0) - f'(0) = 0$.

By part b, we can conclude that $g(x) = 0$ for all x, so that $f(x) = f(0)\cos(x) + f'(0)\sin(x)$, as claimed.

4.1.44 Let V be the space of all matrices S such that $AS = S\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$. Let's denote the column vectors of S by $\vec{u}, \vec{v}$ and $\vec{w}$. The condition $AS = S\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$ means that $A\vec{u} = \vec{u}, A\vec{v} = \vec{v}$ and $A\vec{w} = \vec{0}$. This in turn means that the vectors $\vec{u}$ and $\vec{v}$ have to be on the plane V, while $\vec{w}$ is perpendicular to V. If we choose a basis $\vec{v}_1, \vec{v}_2$ of V and a nonzero vector $\vec{v}_3$ perpendicular to V, then we can write $\vec{u} = a\vec{v}_1 + b\vec{v}_2, \vec{v} = c\vec{v}_1 + d\vec{v}_2, \vec{w} = e\vec{v}_3$, and

$$S = [\vec{u}\ \vec{v}\ \vec{w}] = [a\vec{v}_1 + b\vec{v}_2\ c\vec{v}_1 + d\vec{v}_2\ e\vec{v}_3] = a[\vec{v}_1\ \vec{0}\ \vec{0}] + b[\vec{v}_2\ \vec{0}\ \vec{0}] + c[\vec{0}\ \vec{v}_1\ \vec{0}] + d[\vec{0}\ \vec{v}_2\ \vec{0}] + e[\vec{0}\ \vec{0}\vec{v}_3].$$

Thus $\dim(V) = 5$; the five matrices in the linear combination above form a basis of V.

4.1.12 Yes, the set W of all arithmetic sequences is a subspace of V. Use the fact that a sequence $(x_0, x_1, x_2, \ldots)$ is arithmetic if $x_n = x_0 + kn$ for some constant k.

- The sequence $(0, 0, 0, \ldots)$ is an arithmetic sequence, with $k = 0$.
- If (x_n) and (y_n) are arithmetic sequences (with $x_n = x_0 + pn$ and $y_n = y_0 + qn$), then $x_n + y_n = x_0 + y_0 + (p+q)n$, so that $(x_n + y_n)$ is an arithmetic sequence as well.
- If (x_n) is an arithmetic sequence (with $x_n = x_0 + pn$) and k is an arbitrary constant, then $kx_n = kx_0 + (kp)n$, so that (kx_n) is an arithmetic sequence as well.

4.1.13 Not a subspace: $(1, 2, 4, 8, \ldots)$ and $(1, 1, 1, 1, \ldots)$ are both geometric sequences, but their sum $(2, 3, 5, 9, \ldots)$ is not, since the ratios of consecutive terms fail to be equal, for example, $\frac{3}{2} \neq \frac{5}{3}$.

4.1.14 Yes

- $(0, 0, 0, \ldots, 0, \ldots)$ converges to 0.
- If $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 0$, then $\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n = 0$.
- If $\lim_{n\to\infty} x_n = 0$ and k is any constant, then $\lim_{n\to\infty}(kx_n) = k \lim_{n\to\infty} x_n = 0$.

4.1.15 The set W of all square-summable sequences is a subspace of V:

- The sequence $(0, 0, 0, \ldots)$ is in W.
- Suppose (x_n) and (y_n) are in W. Note that the inequality $(x_n + y_n)^2 \leq 2x_n^2 + 2y_n^2$ holds for all n, since $2x_n^2 + 2y_n^2 - (x_n + y_n)^2 = x_n^2 + y_n^2 - 2x_n y_n = (x_n - y_n)^2 \geq 0$. Thus $\sum_{n=1}^{\infty}(x_n + y_n)^2 \leq 2\sum_{n=1}^{\infty} x_n^2 + 2\sum_{n=1}^{\infty} y_n^2$ converges, so that the sequence $(x_n + y_n)$ is in W as well.
- If (x_n) is in W $\left(\text{so that } \sum_{n=1}^{\infty} x_n^2 \text{ converges}\right)$, then (kx_n) is in W as well, for any constant k, since

$$\sum_{n=1}^{\infty}(kx_n)^2 = k^2 \sum_{n=1}^{\infty} x_n^2$$

will converge.

Ch 4.TF.2 T; check with Definition 4.1.3c.

Ch 4.TF.18 F; For any matrix A, the space of matrices commuting with A is at least two-dimensional. Indeed, if A is a scalar multiple of I_2, then A commutes with all 2×2 matrices, and if A fails to be a scalar multiple of I_2, then A commutes with the linearly independent matrices A and I_2.