FUNCTION SPACES  

Math 21b, O. Knill

Homework: Section 4.1: 6-8,9-11,36,48,58,44*,12*-15*

FROM VECTORS TO FUNCTIONS AND MATRICES. Vectors can be displayed in different ways:

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1  2
3  4
5  6
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The values $(i, e_i)$ can be interpreted as the graph of a function $f : 1, 2, 3, 4, 5, 6 \to \mathbb{R}$, where $f(i) = e_i$.

Also matrices can be treated as functions, but as a function of two variables. If $M$ is a $8 \times 8$ matrix for example, we get a function $f(i, j) = [M]_{ij}$ which assigns to each square of the $8 \times 8$ checkerboard a number.

LINEAR SPACES. A space $X$ which contains 0, in which we can add, perform scalar multiplications and where basic laws like commutativity, distributivity and associativity hold, is called a linear space.

BASIC EXAMPLE. If $A$ is a set, the space $X$ of all functions from $A$ to $R$ is a linear space. Here are three important special cases:

EUCLIDEAN SPACE: If $A = \{1, 2, 3, ..., n\}$, then $X$ is $R^n$ itself.

FUNCTION SPACE: If $A$ is the real line, then $X$ is the space of all functions in one variable.

SPACE OF MATRICES: If $A$ is the set

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(1,1) (1,2) ... (1,m)
(2,1) (2,2) ... (2,m)
  ...
(n,1) (n,2) ... (n,m).
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Then $X$ is the space of all $n \times m$ matrices.

EXAMPLES

- The $n$-dimensional space $R^n$.
- Linear subspaces of $R^n$ like the trivial space $\{0\}$, lines or planes $e^\perp$.
- $M_n$, the space of all square $n \times n$ matrices.
- $P_n$, the space of all polynomials of degree $n$.
- The space $P$ of all polynomials.
- $C^\infty$, the space of all smooth functions on the line.
- $C^0$, the space of all continuous functions on the line.
- $C^\infty(R^3, R^3)$ the space of all smooth vector fields in three dimensions.
- $C^1$, the space of all differentiable functions on the line.
- $C^\infty(R^3)$ the space of all smooth functions in space.
- $L^2$ the space of all functions for which $\int_\infty^\infty f(x) \, dx < \infty$.

ZERO VECTOR. The function $f$ which is everywhere equal to 0 is called the zero function. It plays the role of the zero vector in $R^n$. If we add this function to an other function $g$ we get $0 + g = g$.

Careful, the roots of a function have nothing to do with the zero function. You should think of the roots of a function like as zero entries of a vector. For the zero vector, all entries have to be zero. For the zero function, all values $f(x)$ are zero.

CHECK: For subsets $X$ of a function space, of for a subset of matrices $R^n$, we can check three properties to see whether the space is a linear space:

1) if $x, y$ are in $X$, then $x + y$ is in $X$.
2) If $x$ is in $X$ and $\lambda$ is a real number, then $\lambda x$ is in $X$.
3) $0$ is in $X$.

WHICH OF THE FOLLOWING ARE LINEAR SPACES?

- The space $X$ of all polynomials of the form $f(x) = ax^2 + bx^3 + cx^4$
- The space $X$ of all continuous functions on the unit interval $[-1, 1]$ which are zero at $-1$ and $1$. It contains for example the function $f(x) = x^2 - |x|$.
- The space $X$ of all smooth periodic functions $f(x) = \sin(x)$.
- The space $X = \sin(x) + C^\infty(R)$ of all smooth functions $f(x) = \sin(x) + g$, where $g$ is a smooth function.
- The space $X$ of all smooth functions on $R$ which satisfy $f(1) = 1$. It contains for example $f(x) = 1 + \sin(x) + x$.
- The space $X$ of all smooth functions on $R$ which satisfy $f(2) = 0$ and $f(10) = 0$.
- The space $X$ of all smooth functions on $R$ which satisfy $\lim_{|x| \to \infty} f(x) = 0$.
- The space $X$ of all continuous functions on $R$ which satisfy $\lim_{|x| \to \infty} f(x) = 1$.
- The space $X$ of all smooth functions on $R$ of compact support: for every $f$, there exists an interval $I$ such that $f(x) = 0$ outside that interval.
- The space $X$ of all smooth functions on $R^2$.

If you have taken multivariable calculus you might like the following examples:

- The space $X$ of all vector fields $(P, Q)$ in the plane, for which the curl $Q_x - P_y$ is zero everywhere.
- The space $X$ of all vector fields $(P, Q, R)$ in space, for which the divergence $P_x + Q_y + R_z$ is zero everywhere.
- The space $X$ of all vector fields $(P, Q)$ in the plane for which the line integral $\int_C F \cdot dr$ along the unit circle is zero.
- The space $X$ of all vector fields $(P, Q, R)$ in space for which the flux through the unit sphere is zero.
- The space $X$ of all functions $f(x, y)$ of two variables for which $\int_0^1 \int_0^1 f(x, y) \, dx \, dy = 0$. 