CHECKLIST FINAL (since 2. Midterm), Math21b, O.Knill

Abstract

DIFFERENTIAL EQUATIONS $\dot{x}(t) = F(x(t))$ define dynamical systems. For linear differential equations $\dot{x} = Ax$ the solution is the DISCRETE DYNAMICAL SYSTEM $x(t+1) = Bx(t) = e^A x(t)$ but this is not convenient to compute. LINEAR DIFFERENTIAL EQUATIONS are written as $\dot{x} = Ax$ of $p(D)f = g$. In the second case, we factor the polynomial $p(\lambda) = \prod_i (\lambda - \lambda_i)$ to get the homogeneous solution $f(x) = \sum_i a_i e^{\lambda_i x}$ and look then for a special solution. In the more general case $\dot{x} = Ax$ the system solved it by DIAGONALIZATION. Each eigenvector v_k satisfying $Av_k = \lambda_k v_k$ and evolves like $v_k(t) = e^{\lambda_k t} v_k$. A general initial condition $x = \sum_k a_k v_k$ evolves then like $x(t) = \sum_k a_k e^{\lambda_k t} v_k$. The same procedure solves PARTIAL DIFFERENTIAL EQUATIONS like the HEAT $\dot{f} = D^2 f$ or WAVE EQUATION $\ddot{f} = D^2 f$, where FOURIER diagonalizes D^2.

Glossary

COMPLEX NUMBERS $x + iy = re^{i\theta}$.
LINEAR DISCRETE DYNAMICAL SYSTEM Linear map $x \mapsto Ax$ defines orbit $\vec{x}(t+1) = A\vec{x}(t)$.
ASYMPTOTIC STABILITY FOR DISCRETE SYSTEMS $A^n \vec{x} \to 0$ for all $\vec{x}$.
LINEAR SPACE X If x, y are in X, then $x + y, \lambda x$ are in X. Especially, 0 is in X.
LINEAR MAP $T(x + y) = T(x) + T(y), T(\lambda x) = \lambda T(x)$. Especially, $T(0) = 0$.
DIAGONALIZATION possible if A is symmetric or all eigenvalues are different.
TRACE. $\text{tr}(A) =$ sum of diagonal entries, $\sum_j \lambda_j$.
DETERMINANT. $\det(A) =$ product of diagonal entries, $\prod_j \lambda_j$.
TRACE AND DETERMINANT. Determine stability in two dimensions.
LINEAR DIFFERENTIAL EQUATION $\dot{x} = Ax$, where A is a matrix.
DIFFERENTIAL OPERATOR polynomial in D. Example $T = p(D) = D^2 + 3D, Tx = x'' + 3x'$.
HOMOGENEOUS DIFFERENTIAL EQUATION $p(D)f = 0$. Example: $f'' + 3f' = 0$.
INHOMOGENEOUS DIFFERENTIAL EQUATION $p(D)f = g$. Example: $f'' + 3f' = \sin(t)$.
1D LINEAR DIFFERENTIAL EQUATION $f' = \lambda f$, $f(t) = e^{\lambda t} f(0)$.
1D HARMONIC OSCILLATOR $f'' = c^2 f$, $f(t) = f(0)\cos(ct) + f'(0)\sin(ct)/c$.
LINEAR ODE WITH CONSTANT COEFFICIENTS $p(D)f = g$.
GENERALIZED INTEGRATION $((D - \lambda)^{-1} f)(x) = e^{\lambda x}(\int_0^x f(t)e^{-\lambda t}\, dt + C)$.
HOMOGENEOUS LINEAR ODE $p(D)f = 0$.
INNER PRODUCT $(f, g) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\, dx$.
LENGTH (square) $(f, f) = |f|^2$.
FOURIER SERIES $f(x) = a_0/\sqrt{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$.
FOURIER BASIS $1/\sqrt{2}, \cos(nx), \sin(x)$ for 2π-periodic functions.
FOURIER COEFFICIENTS $a_0 = (f, 1/\sqrt{2})$, $a_n = (f, \cos(nx))$, $b_n = (f, \sin(nx))$.
PARSEVAL $a_0^2/2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = |f|^2$, if $f(x) = a_0/\sqrt{2} + \sum_{n\geq 1} a_n \cos(nx) + b_n \sin(nx)$.
HEAT EQUATION $\dot{f} = \mu D^2 f$ with solution $f(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx)e^{-n^2\mu t}$.
WAVE EQUATION $\ddot{f} = c^2 D^2 f$ with solution $f(x,t) = \sum_{n=1}^{\infty} a_n \sin(nx)\cos(nct) + \frac{b_n}{nc}\sin(nx)\sin(nct)$
STABILITY FOR DISCRETE 2×2 **SYSTEMS**: $|\lambda_i| < 1$.
STABILITY FOR CONTINUOUS 2×2 **SYSTEMS**: $\text{Re}(\lambda_i) < 0$.
NONLINEAR DIFFERENTIAL EQUATION $\dot{x} = f(x, y), \dot{y} = g(x, y)$.
EQUILIBRIUM POINTS points, where $f(x, y) = g(x, y)$.
NULLCLINES are curves, where $f(x, y) = 0$ or $g(x, y) = 0$.
JACOBEAN $\begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix}$ at critical point (x_0, y_0) of $\dot{(x)} = f(x, y), \dot{y} = g(x, y)$.

Skills checklist

UNDERSTAND LINEAR SPACES, LINEAR MAPS.
SOLVE DISCRETE DYNAMICAL SYSTEMS $x(n + 1) = Ax(n)$. By diagonalization.
SOLVE CONTINUOUS DYNAMICAL SYSTEMS $\dot{x} = Ax$. By diagonalization.
SOLVE DIFFERENTIAL EQUATIONS $p(D)f = g$ by factoring p, homogeneous and inhomogeneous case
ASYMPTOTIC STABILITY for discrete dynamical systems.
ASYMPTOTIC STABILITY for continuous dynamical systems.
PLOT PHASE SPACE for nonlinear systems: equilibrium points, nullclines, nature of equilibrium points.
MATCH PHASE SPACE WITH SYSTEM.
MAKE FOURIER SYNTHESIS of function $f(x)$ on $[-\pi, \pi]$.
DIAGONALIZATION of D^2 by Fourier basis.
SOLVE HEAT EQUATION with given initial condition by diagonalization.
SOLVE WAVE EQUATION with given initial condition by diagonalization.

The problems of the review lecture.

1) LINEAR OR NOT? (Quiz)

Linear space or not?
2) All smooth functions on $[0, 2\pi]$ satisfying $\int_0^{2\pi} f(x)\, dx = 0$.
3) All smooth functions satisfying $f(10) = 0$
5) All Smooth functions on the line satisfying $f'(10) = 10$.
7) All symmetric 2×2 matrices.
11) All polynomials of degree 10.

Linear map or not?
2) $T(f)(x) = x^2 f(x)$.
3) $T(f)(x) = f''(x)$.
5) $T(f)(x) = f(1)^2 + f(x)$.
7) $T(f)(x) = f(5)$.
11) $T(f)(x) = f(x)f'(x)$.

2) DISCRETE DYNAMICAL SYSTEMS.

Solve the initial value problem problem

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ x - y \end{bmatrix}$$

with initial condition $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

3) DIFFERENTIAL EQUATIONS. Determine the nature and stability of the systems $\dot{x} = Ax$:

$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$, $A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$.

4) DIFFERENTIAL EQUATIONS $p(D)f = g$. Solve the initial value problem for the following ordinary differential equations. For real λ, use the formula $(D - \lambda)^{-1} f$.

$$f' - 3f = e^t .$$

$$f'' - 6f' + 9f = e^t .$$

$$f'' + 9f = e^t .$$

$$f'' + 6f' + 8f = t .$$

5) NONLINEAR DYNAMICAL SYSTEM

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^2 + xy - 3x \\ xy - y^2 + y \end{bmatrix} . \qquad \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ y^2 + x^2 - 1 \end{bmatrix} .$$

6) FOURIER SERIES.

Find the Fourier Series of $f(x) = \cos^2(x) + 5x + 7 + \sin(3x)$.

7) HEAT EQUATION

Solve the heat equation $f_t = f_x$ on $[0, \pi]$ with $f(x, 0) = 5x + \sin(3x)$

8) WAVE EQUATION

Solve the wave equation $f_{tt} = f_{xx}$ on $[0, \pi]$ with $f(x, 0) = 5x + \sin(3x)$ and $f_t(x, 0) = \sin(17x)$.

Words of wisdom:

- Repetition is the mother of all skill.
- Odd functions have sin-Fourier series.
- If you understand what you're doing, you're not learning anything.
- The columns of a matrix are the images of the standard basis vectors.
- Fourier coefficients are coordinates of a function in a special basis.
- A laugh can eliminate a thousand worries.