Homework for Tuesday May 7, 1*,2,3,4*,5,6* in Section 10.4

LAPLACE EQUATION. The linear map $T \mapsto \Delta T = T_{xx} + T_{yy}$ for smooth functions in the plane is called the **Laplacian**. The equation $\Delta T = 0$ is called the **Laplace equation**.
If T satisfies the Laplace equation, it is in the kernel of Δ. One calls such functions **harmonic**. For example $T(x,y) = x^2 - y^2$ is a harmonic function in the plane. More generally, $T(x,y) = \text{Re}((x+iy)^n$ or $T(x,y) = \text{Im}((x+iy)^n)$ are harmonic.

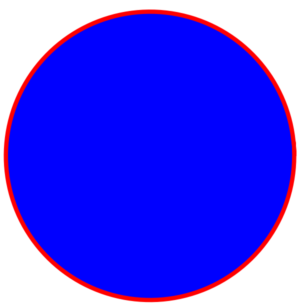
DIRICHLET PROBLEM: find a function $T(x,y)$ defined in a region Ω of the plane which satisfies $\Delta T(x,y) = 0$ inside the region and takes prescribed values $T_0(x,y)$ at the boundary. This boundary value problem can be solved explicitely in some cases.

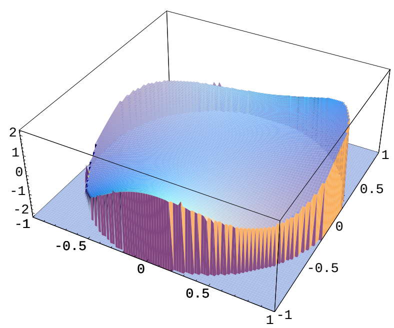
IN A DISC $\{x^2 + y^2 \leq 1\}$. If $f(t) = T(\cos(t), \sin(t))$ is prescribed on the boundary and T satisfies $\Delta T = 0$ inside the disc, then T can be found via Fourier theory:

If $f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt)$, then $T(x,y) = a_0 + \sum_n a_n \text{Re}((x+iy)^n) + b_n \text{Im}((x+iy)^n)$ satisfies the Laplace equation and $T(\cos(t), \sin(t)) = f(t)$ on the boundary of the disc.

PROOF. The general case is a sum of the following cases:
if $f = a_0$ is constant on the boundary then $T(x,y) = a$,
if $f(t) = \sin(nt)$ on the boundary $z = e^{it} = x + iy$ then $T(x,y) = \text{Im}(z^n)$ and if $f(t) = \cos(nt)$ on the boundary then $T(x,y) = \text{Re}(z^n)$ is a solution.

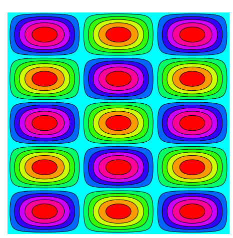
For example $\text{Re}(z^2) = x^2 - y^2 + i(2xy)$ gives the two harmonic functions $f(x,y) = x^2 - y^2$ and $g(x,y) = 2xy$: they satify $(\partial_x^2 + \partial_y^2) f = 0$.

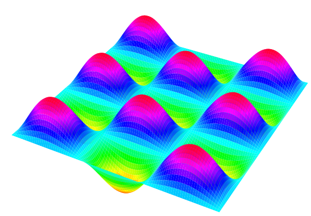




ESPECIALLY: $T(0,0) = \frac{1}{2}\int_0^{2\pi} f(t)\, dt$. This is called the **mean value property** of harmonic functions.

IN A SQUARE. The Laplace equation in a square $[0,\pi]^2$ is solved via Fourier theory also: If $T(x,y) = \sum_{k,m} a_{n,m} \sin(nx)\sin(my)$, then $T_{xx} + T_{yy} = \sum_{n,m} -(n^2+m^2) a_{n,m}$. You see that $\lambda = -(n^2+m^2)$ are the eigenvalues of Δ on the square.





PHYSICAL RELEVANCE.
- If the **temperature** distribution on the boundary of a region Ω, then $T(x,y)$ is the stable temperature distribution inside that region.

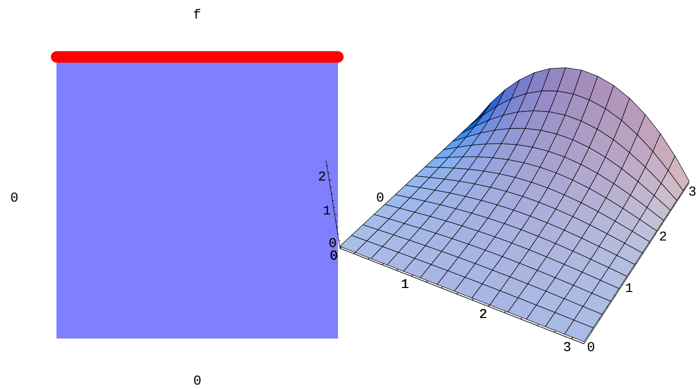
- If the **potential** of an electric charge distribution is given on the boundary of Ω and no charge is inside the region, then $U(x,y)$ satisfies $\Delta U = 0$. The electric field inside Ω is then ∇U.

An other application is that the velocity v of an ideal incompressible fluid satisfies $v = \nabla U$, where U satisfies the Laplace differential equation $\Delta U = 0$ in the region.

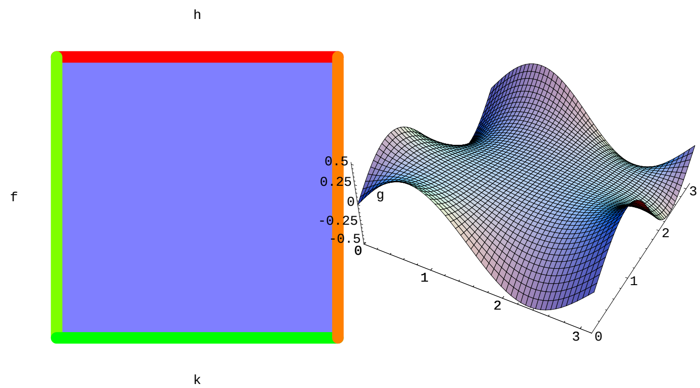
EXAMPLE: SQUARE.

We solve first the case, when T vanishes on three sides $x = 0, x = \pi, y = 0$ and $T(x,\pi) = f(x)$. Seperation of variables gives then $T(x,y) = \sum_n a_n \sin(nx)\sinh(ny)$, where the coefficients a_n are obtained from $T(x,\pi) = \sum_n b_n \sin(nx) = \sum_n a_n \sin(nx)\sinh(n\pi)$, where $b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\ dx$ are the Fourier coefficients of f. The solution is therefore

$$T_f(x,y) = \sum_{n=1}^{\infty} b_n \sin(nx)\sinh(ny)/\sinh(n\pi)$$



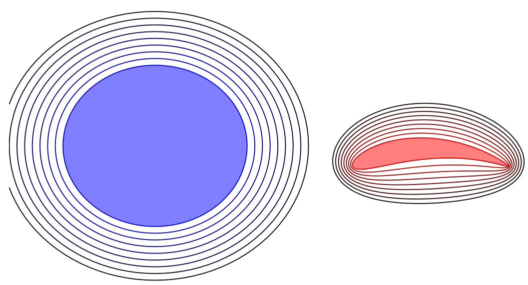
Solutions to general boundary conditions can be obtained by adding the solutions in the following cases: $T_f(x,\pi-y)$ solves the case, when T vanishes on the sides $x = 0, x = \pi, y = \pi$ and is f on $y = 0$. The function $T_f(y,x)$ solves the case, when T vanishes on the sides $y = 0, y = \pi, x = 0$ and $T_f(\pi - y, x)$ solves the case, when T vanishes on $y = 0, y = \pi$ and $x = \pi$. The general solution is $T(x,y) = T_f(x,y) + T_g(x,\pi - y)$ $+T_h(y,x) + T_k(\pi - y, x)$



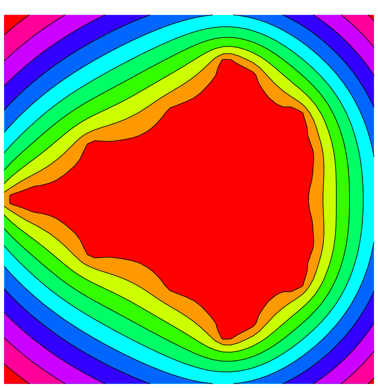
A GENERAL REGION.

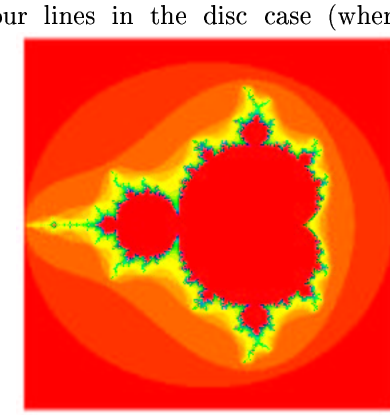
For a general region, one uses numerical methods. One possibility is by **conformal transportation**: to map the region into the the disc using a complex map F, which maps the region Ω into the disc. Solve then the problem in the disc with boundary value $T(F^{-1}(x,y))$ on the disc. If $S(x,y)$ is the solution there, then $T(x,y) = S(F^{-1}(x,y))$ is the solution in the region Ω.

The picture shows the example of the Joukowski map $z \mapsto (z + 1/z)/2$ which has been used in fluid dynamics (N.J. Joukowski (1847-1921) was a Russian airodynamics researcher.)



COMPLEX DYNAMICS. Also in complex dynamics, harmonic functions appear. Finding properties of complicated sets like the **Mandelbrot set** is done by mapping the exterior to the outside of the unit circle. If the Mandelbrot set is charged then the contour lines of equal potential can be obtained as the corresponding contour lines in the disc case (where the lines are circles).





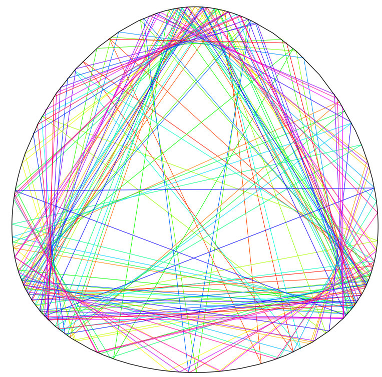
A tourist view on topics in PDE's:

POISSON EQUATION. The equation $\Delta T = \lambda T$ is the eigenvalue problem for the Laplacian. The equation $\Delta T = S$ is called the **Poisson equation**. It is important in **electrostatics**, for example to determine the electromagnetic field when the charge and current distribution is known: $\Delta U(x) = -(1/\epsilon_0)\rho$, then $E = \nabla U$ is the electric field to the charge distribution $\rho(x)$.

PARTICLE IN A REGION. If we have a bounded region Ω, we can look at all smooth functions which are zero on the boundary of Ω. The possible eigenvalues of $\Delta f = \lambda f$ describe the possible energies of a particle in Ω.

QUANTUM CHAOS.

A research topic is to understand the eigenvalues and eigenvectors of the Laplacian in a bounded region Ω acting on smooth functions with zero boundary conditions. If one has no closed formulas for for the eigenvalues λ_n and eigenfunctions f_n some people call this **quantum chaos**. For convex regions (regions, where the connection between two points of the region are inside the region), there are relations known between the eigenvalue problem $\Delta f = \lambda f$ and the **billiard problem** in the region Ω (see picture). Quantum chaos is believed to be related to **chaos** of the billiard map problem.



OPEN PROBLEM. Assume we we have two smooth convex regions in the plane for which the eigenvalues λ_j of the Laplacian are the same. Can the regions be transfered into each other by a rotation and translation?

SCHRÖDINGER EQUATION. If H is the energy operator, then $i\hbar\dot{f} = Hf$ is called the **Schrödinger equation**. If $H = -\hbar^2\Delta/(2m)$, this looks very much like the heat equation, if there were not the i. If f is an eigenvalue of H, then $i\hbar\dot{f} = \lambda f$ and $f(t) = e^{i\hbar t}f(0)$. In the heat equation, we would get $f(t) = e^{-\mu(t)}f(0)$.

OVERVIEW OVER PDE's. Which differential equations do you have to master for this course?

$T_t = \mu T_{xx}$	heat equation	yes
$T_{tt} = c^2 T_{xx}$	wave equation	yes
$\Delta f = 0$	Laplace equation	yes
$T_t = cT_x$	transport equation	no
$\Delta f = g$	Poisson equation	no
$T_t = \mu\Delta f$	heat equation in 2D/3D	no
$T_{tt} = c^2\Delta f$	wave equation in 2D/3D	no
$i\hbar\dot{f} = (\hbar^2/(2m))\Delta f + Vf$	Schrödinger equation	no

Outlook from this linear algebra course: You might see objects from this course in other subjects:

1) **Graph theory:** Graphs can be studied with matrices. The adjecency matrix of a graph encodes the graph. The paths of length n in a graph can be read of from the n'th power A^n.

2) **Control theory:** is formulated in terms of matrices. This has practical applications like to stabilize the flight of helicopters and airoplanes.

3) **Inverse problems:** the use of Fourier theory or integral equations can help to understand the hidden structure of crystals. Linear algebra is used in tomography.

4) **Coding theory:** codes are often linear spaces. You have seen an example of a code in the context of error correction.

5) **Dynamical system theory:** the notion of chaos and stability is defined through matrices. A discrete dynamical system $x \mapsto f(x)$ is chaotic if the Jacobian $Df^n(x)$ of the n'th iterate grows exponentially on a large set.

6) **Graphics:** cubic splines, or quadric patches for surfaces are determined through linear maps. Rotations, translations, scaling transformations are basic building blocks for ray tracing software or CAD applications.

7) **Markov chain:** finding the equilibrium situation of a linear random process is an eigenvalue problem. You have seen this in the example of the pollution of the three lakes.

8) **Chemistry**. Spectra of atoms or molecules: the spectrum consists of eigenvalues of a linear map.

9) **Partial differential equations:** heat or wave equation are only two examples. The Navier Stokes equations (a nonlinear equation) is important in weather prediction, the Schrödinger equation in quantum mechanics etc.

10) **Neural nets:** are formulated in terms of matrices.

11) **Game theory:** games are often represented by Pay-Off Matrices.

12) **Data fitting:** least square fitting is related to the projection onto a lower dimensional space.

13) **Data compression:** Compressions like MP3, MPEG, JPG, rely on discrete Fourier transforms.

14) **Optimization:** extremal problems under constraints lead to linear systems.

15) **Quantum mechanics:** observables in quantum mechanics are linear maps. Spectra and eigenvalues are have real physical meaning.

16) **Biology:** in population dynamics, differential equations can allow predictions of population growths.

17) **General relativity:** relies on a new geometry where the dot product depends on time. Tensors are generalisations of matrices.

18) **Geometry:** the study of manifolds (a notion which generalizes the notion of surfaces) is done with different tools, with multilinear algebra or partial differential equations.

19) **Group theory.** Groups are sets in which one can reasonably "multiply" and invert. An example are all invertible nxn matrices or all orthogonal nxn matrices. Whenever you realize a group as a set of matrices, this is called a "representation". Representations are important in physics: **quarks** for examples can be understood through representations of certain 3x3 matrices.

20) **Number theory.** Linear theory pops up in number theory. Laplace transform, generating functions or Dirichlet series are used in a similar way as Fourier theory for solving problems.

21) **Cryptology.** Some algorithms like the "Linear Sieve factorisation algorithm" in computational number theory rely on linear algebra.

22) **Brownian motion.** This is not only relevant in thermodynamics but also in financial mathematics or thermodynamics. Brownian paths are elements in the space C of continuous paths.

Final words:

```
The science of mathematics presents the most brilliant example of how
pure reason may successfully enlarge its domain without the aid of
experience.
                                  -- E. Kant

There is no branch of mathematics, however abstract, which may not some
day be applied to phenomena of the real world.
                                  -- N. Lobatchevsky

Mathematics consists of proving the most obvious thing in the least
obvious way.
                                  -- G. Polya

It is still an unending source of surprise for me to see
how a few scribbles on a blackboard or on a sheet of paper
could change the course of human affairs.
                                  -- Stanislaw Ulam.
```