Fourier Series 5/30/2002 **Math 21b, O. Knill**

Homework for Thursday Mai 3, 1,2,3,4*,5*,6* in Section 10.2

HOW DOES IT WORK? Continuous functions $f(x)$ on $[-\pi,\pi]$ form a linear space $C([-\pi,\pi]$. The numbers $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}$ are called the **Fourier coefficients** of f. If f is piecewise differentiable, we can recover f from these coefficients by $f(x) = \sum_k c_k e^{ikx}$. When we write $f = f_{even} + f_{odd}$, where $f_{even}(x) = (f(x)+f(-x))/2$ and $f_{odd}(x) = ((f(x) - f(-x))/2$, then $f_{even}(x) = a_0/\sqrt{2} + \sum_{k=1} a_k \cos(kx)$ and $f_{odd}(x) = \sum_{k=1}^{\infty} b_k \sin(kx)$. The sum of the cos-**series** for f_{even} and the sin-**series** for f_{odd} up gives the **real Fourier series** of f. Those coefficients can be obtained as $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx, a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)/\sqrt{2}\,dx$, $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx$.

EXAMPLE 1. Let $f(x) = x$ on $[-\pi,\pi]$. This is an odd function ($f(-x)+f(x) = 0$) so that it has a sin series: with $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(kx)\,dx = \frac{1}{\pi}(x\cos(kx)/k + \sin(kx)/k^2|_{-\pi}^{\pi}) = 2(-1)^{k+1}/k$ we get $x = \sum_{k=1}^{\infty} 2\frac{(-1)^{k+1}}{k}\sin(kx)$. For example, $\pi/2 = 2(1 - 1/3 + 1/5 - 1/7...)$ a **formula of Leibnitz**.
EXAMPLE 2. Let $f(x) = \cos(x) + 1/7\cos(5x)$. This **trigonometric polynomial** is already the Fourier series! The nonzero coefficients are $a_1 = 1, a_5 = 1/7$.
EXAMPLE 3. Let $f(x) = |x|$ on $[-\pi,\pi]$. This is an even function $f(-x) - f(x) = 0$ so that it has a cos series: with $a_0 = 1/(2\sqrt{2}), a_k = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos(kx)\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos(kx)\,dx = \frac{2}{\pi}(+x\sin(kx)/k + \cos(kx)/k^2|_0^{\pi}) = \frac{2}{\pi}((-1)^k - 1)/k^2$ for $k > 0$.

WHERE ARE FOURIER SERIES USEFUL? Examples:

- **Partial differential equations**. PDE's like $\ddot{u} = c^2u''$ become ODE's: $\ddot{u}_k = c^2k^2u_k$ can be solved as $u_k(t) = a_k\sin(ckt)u_k(0)$ and lead to solutions $u(t,x) = \sum_k a_k\sin(ckt)\sin(kx)$ of the PDE.
- **Sound** Coefficients a_k form the **frequency spectrum** of a sound f. **Filters** supress frequencies, **equalizers** transform the Fourier space, **compressors** (i.e.MP3) select frequencies relevant to the ear.
- **Analysis:** $\sum_k a_k\sin(kx) = f(x)$ give explicit expressions for sums which would be hard to evaluate otherwise. The Leibnitz sum $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + ...$ (Example 1 above) is an example.
- **Number theory:** Example: if α is irrational, then the fact that $n\alpha$ (mod1) are uniformly distributed in $[0,1]$ can be understood with Fourier theory.
- **Chaos theory:** Quite many notions in Chaos theory can be defined or analyzed using Fourier theory. Examples are mixing, weak mixing, ergodicity.
- **Quantum dynamics:** Transport properties of materials are related to spectral questions for their Hamiltonians. The relation is given by Fourier theory.
- **Crystallography:** X ray Diffraction patterns of a crysital, analysed using Fourier theory reveal the structure of the crystal.
- **Probability theory:** The Fourier transform $\chi_X = E[e^{iX}]$ of a random variable is called **characteristic function**. Independent case: $\chi_{x+y} = \chi_x\chi_y$.
- **Image formats:** like JPG compress by cutting irrelevant parts in Fourier space.

WHY DOES IT WORK? One has a **dot product** on functions with $f \cdot g = \frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{f}(x)g(x)\,dx$. The functions $e_k = e^{ikx}$ serve as **basis vectors**. As in Euclidean space, where $v \cdot e_k = v_k$ is the **k'th coordinate** of v, the Fourier coefficient $c_k = e_k \cdot f = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx}f(x)\,dx$ is the k-th coordinate of f. In the same way as we wrote $v = \sum_k v_k e_k$ in Euclidean space, we have $f(x) = \sum_k c_k e_k(x)$. The vectors e_k are orthonormal: $\frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{e_k}(x)e_m(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(m-k)ix}\,dx$ which is zero for $m \neq k$ and 1 for $m = k$.

> A Fourier series is an expansion of the "vector" f with respect to the orthonormal basis $\{e_k\}_{k=-\infty}^{\infty}$. The Fourier coefficients are the coordinates of f in that basis.

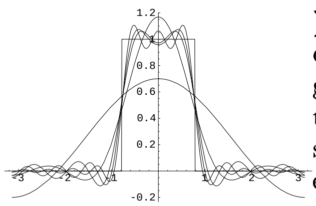
One could choose an other basis. **Wavelets** expansions are examples different from Fourier series.

LENGTH, DISTANCE AND ANGLE. With a dot product, one has a **length** $||f||^2 = f \cdot f = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx$ and can measure **distances** between two functions as $||f - g||$ and **angles** between functions f, g by $\cos(\alpha) = f \cdot g/(||f||||g||)$. Functions are called **orthogonal** if $(f,g) = f \cdot g = 0$. For example, an even function f and an odd function g are orthogonal. The functions e^{inx} form an orthonormal family. The vectors $\sqrt{2}\cos(kx), \sqrt{2}\sin(kx)$ form an orthonormal family, $\cos(kx)$ is in the linear space of **even functions** and $\sin(kx)$ in the linear space of **odd functions**. If we work with sin or cos Fourier series, we use the dot product $\boxed{f \cdot g = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx}$ for which **the functions** $1/\sqrt{2}, \cos(kx), \sin(kx)$ **are orthonormal**.

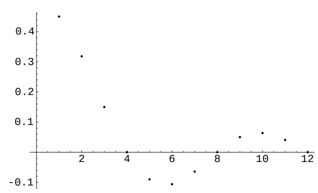
REWRITING THE DOT PRODUCT. If $f(x) = \sum_k c_k e^{ikx}$ and $g(x) = \sum_k d_k e^{ikx}$, then $\frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(x)} g(x)\ dx = \frac{1}{2\pi}\int \sum_{k,m} \overline{c}_k e^{-ikx} d_m e^{imx} = \sum_k \overline{c_k} d_k$. The dot product is the sum of the product of the coordinates as in finite dimensions. If f and g are even, then $f = f_0/\sqrt{2} + \sum_{k=1}^{\infty} f_k \cos(kx), g = g_0/\sqrt{2} + \sum_k g_k \cos(kx)$ and $\frac{1}{\pi}\int_{\pi}^{\pi} f(x)g(x)\ dx = \sum_{k=0}^{\infty} f_k g_k$. If f and g are odd, and $f = \sum_k f_k \sin(kx)$, $g = \sum_{k=1}^{\infty} g_k \sin(kx)$ then $(f,g) = f \cdot g = \sum_{k=1}^{\infty} f_k g_k$. Especially $\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2 = \sum_{k=1}^{\infty} f_k^2$. This gives formulas for series.

EXAMPLE. $f(x) = x = 2(\sin(x) - \sin(2x)/2 + \sin(3x)/3 - \sin(4x)/4 + ...$ has coefficients $f_k = 2(-1)^{k+1}/k$ and so $4(1 + 1/4 + 1/9 + ...) = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\ dx = 2\pi^2/3$ or $1 + 1/4 + 1/9 + 1/6 + ...) = \pi^2/6$.

APPROXIMATIONS.

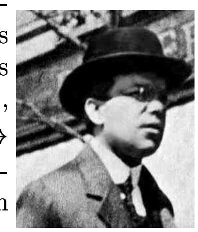


If $f(x) = \sum_k b_k \cos(kx)$, then $f_n(x) = \sum_{k=1}^{n} b_k \cos(kx)$ is an approximation to f. Because $||f - f_k||^2 = \sum_{k=n+1}^{\infty} b_k^2$ goes to zero, the graphs of the functions f_n come for large n close to the graph of the function f. The picture to the left shows an approximation of a piecewise continuous even function, the right hand side the values of the coefficients $a_k = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\ dx$.



SOME HISTORY. The **Greeks** approximation of planetary motion through **epicyles** was an early use of Fourier theory: $z(t) = e^{it}$ is a circle (Aristarchus system), $z(t) = e^{it} + e^{int}$ is an epicycle (Ptolemaeus system), **18'th century** Mathematicians like Euler, Lagrange, Bernoulli knew experimentally that Fourier series worked.

Fourier's (picture left) claim of the convergence of the series was confirmed in the **19'th century** by Cauchy and Dirichlet. For continuous functions the sum does not need to converge everywhere. However, as the 19 year old Fejér (picture right) demonstrated in his theses in 1900, the coefficients still determine the function $\sum_{k=-(n-1)}^{n-1} \frac{n-|k|}{n} f_k e^{ikx} \to f(x)$ for $n \to \infty$ if f is continuous and $f(-\pi) = f(\pi)$. Partial differential equations, i.e. the **theory of heat** had sparked early research in Fourier theory.

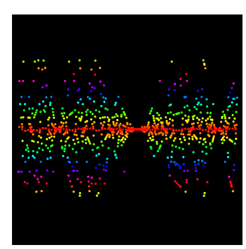


OTHER FOURIER TRANSFORMS. On a finite interval one obaines a series, on the line an integral, on finite sets, finite sums. The discrete Fourier transformation (DFT) is important for applications. It can be determined efficiently by the (FFT=**Fast Fourier transform**) found in 1965, reducing the n^2 steps to $n\log(n)$.

Domain	Name	Synthesis	Coefficients
$\mathbf{T} = [-\pi, \pi)$	**Fourier series**	$f(x) = \sum_k \hat{f}_k e^{ikx}$	$\hat{f}_k = \frac{1}{2\pi}\int_{-\pi} \pi f(x) e^{-ikx}\ dx.$
$\mathbf{R} = (-\infty, \infty))$	**Fourier transforms**	$f(x) = \int_{\infty}^{\infty} \hat{f}(k) e^{ikx}\ dx$	$\hat{f}(k) = \frac{1}{2\pi}\int_{\infty}^{\infty} \hat{f}(k) e^{-ikx}\ dx$
$\mathbf{Z}_n = \{1, .., n\}$	**DFT**	$f_m = \sum_{k=1}^{n} \hat{f}_k e^{imk2\pi/n}$	$\hat{f}_k = \frac{1}{n}\sum_{m=1}^{n} f_m e^{-ikm2\pi/n}\ dx.$

All these transformations can be defined in dimension d. Then $k = (k_1, ..., k_d)$ etc. are vectors. 2D DFT is for example useful in **image manipulation**.

20'th CENTURY FOURIER THEORY. The Fourier series can be defined for functions f for which $\int_{-\pi}^{\pi} |f(x)|\ \ dx$ is defined and finite. Where does the Fourier series converge? Kolmogorov gave an example of such a function, where the Fourier series **diverges everywhere**. In 1964, Carlson cleared up, what happens for continuous functions: while the Fourier series can diverge at some points, it converges for "almost all" x (for random x, the series converges with probability 1). For functions f satisfying $\int_{-\pi}^{\pi} |f(x)|^2\ dx$, the Fourier series converges almost everywhere. For discontinuous functions f like functions f supported on a fractal, the Fourier spectrum is unusual. Properties like **dimension** can sometimes be read of from the Fourier spectrum. The picture shows the Fourier coefficients for a fractal.



COMPUTER ALGEBRA. Packages like Mathematica have the discrete Fourier transform built in $Fourier[\{0.3, 0.4, 0.5\}]$ for example, gives the DFT of a three vector. You can perform a simple Fourier analysis yourself by listening to a sound like $Play[Sin[2000 * x * Floor[7 * x]/12], \{x, 0, 20\}]$. The ear and your brain produces the Fourier transform, when you listen to music.