Nonlinear dynamical systems 4/23/2002	Math 21b, O. Knill

Homework for Thursday April 25, 1,2,3*,4,5 in 9.4 handout

SUMMARY. For linear systems $d/dtx = \dot{x} = Ax$ the eigenvalues of A determine the behavior completely. For nonlinear systems, **chaos** can set in (i.e. the Rössler system), explicit formulas for solutions are no more available in general (not even power series solutions which are valid for all t) and orbits can go off to infinity in finite time like in $\dot{x} = x^2$ which has the solution $x(t) = -1/(t - x(0))$ and reaches for $x(0) = 1$ infinity at time $t = 1$.
Linearity is often too crude. The exponential growth $\dot{x} = ax$ of a bacteria colony for example is slowed down due to lack of food. The **logistic model** $\dot{x} = (a/M)x(M - x)$ is then more accurate, where M is the population size for which bacteria starve so much that the growth has stoped: $x(t) = M$, then $\dot{x}(t) = 0$. Nonlinear systems can be investigated with **qualitative methods**. In 2D: $\dot{x} = f(x, y), \dot{y} = g(x, y)$, where chaos does not happen, the analysis of **equilibrium points** and the **linear approximation** at those points in general allows to understand the system completely. In general, the analysis of equilibrium points and linear approximation at those points is a good starting point and where linear algebra becomes useful.

EQUILIBRIUM POINTS. A vector x_0 in the phase space of the system is an **equilibrium point** of $\dot{x} = f(x)$ if $f(x_0) = 0$. If $x(0) = x_0$ then $x(t) = x_0$ for all times.

JACOBIAN MATRIX. If x_0 is an equilibrium point for $\dot{x} = f(x)$ then $[A]_{ij} = \frac{\partial}{\partial x_j} f_i(x)$ is called the **Jacobian** at x_0. For two dimensional systems

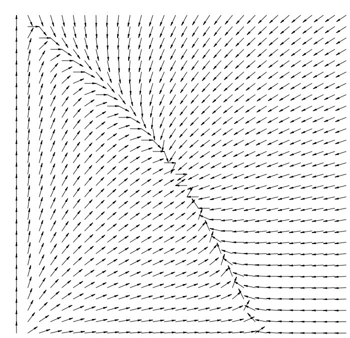
$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned} \qquad \text{this is the } 2x2 \text{ matrix} \qquad A = \begin{bmatrix} \frac{\partial f}{\partial x}(x,y) & \frac{\partial f}{\partial y}(x,y) \\ \frac{\partial g}{\partial x}(x,y) & \frac{\partial g}{\partial y}(x,y) \end{bmatrix}.$$

The linear ODE $\dot{y} = Ay$ with $y = x - x_0$ approximates the nonlinear system well near the equilibrium point. The Jacobian is the linear approximation of $F = (f, g)$ near x_0.

VECTOR FIELD. In two dimensions, we can draw the vector field by hand: attaching a vector $(f(x, y), g(x, y))$ at each point (x, y). To find the equilibrium points it helps to draw the **nullclines** $\{f(x, y) = 0\}, \{g(x, y) = 0\}$. The equilibrium points are located on intersections of nullclines. The eigenvalues of the Jacobeans at equilibrium points allow to draw the vector field near equilibrium points. This information is sometimes enough to draw the vector field **by hand**.

MURRAY SYSTEM (see handout) $\dot{x} = x(6 - 2x - y), \dot{y} = y(4 - x - y)$ has the nullclines $x = 0, y = 0, 2x + y = 6, x + y = 5$. There are 4 equilibrium points $(0, 0), (3, 0), (0, 4), (2, 2)$. The Jacobian matrix of the system at the point (x_0, y_0) is $\begin{bmatrix} 6 - 4x_0 - y_0 & -x_0 \\ -y_0 & 4 - x_0 - 2y_0 \end{bmatrix}$.

Equilibrium	Jacobean	Eigenvalues	Nature of equilibrium
(0,0)	$\begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$	$\lambda_1 = 6, \lambda_2 = 4$	Unstable source
(3,0)	$\begin{bmatrix} -6 & -3 \\ 0 & 1 \end{bmatrix}$	$\lambda_1 = -6, \lambda_2 = 1$	Hyperbolic saddle
(0,4)	$\begin{bmatrix} 2 & 0 \\ -4 & -4 \end{bmatrix}$	$\lambda_1 = 2, \lambda_2 = -4$	Hyperbolic saddle
(2,2)	$\begin{bmatrix} -4 & -2 \\ -2 & -2 \end{bmatrix}$	$\lambda_i = -3 \pm \sqrt{5}$	Stable sink



USING TECHNOLOGY (Example: Mathematica). Plot the vector field:

```
Needs["Graphics`PlotField`"]
f[x_,y_]:={x(6-2x-y),y(5-x-y)};PlotVectorField[f[x,y],{x,0,4},{y,0,4}]
```

Find the equilibrium solutions:

```
Solve[{x(6-2x-y)==0,y(5-x-y)==0},{x,y}]
```

Find the Jacobian and its eigenvalues at $(2, 2)$:

```
A[{x_,y_}]:={{6-4x,-x},{-y,5-x-2y}};Eigenvalues[A[{2,2}]]
```

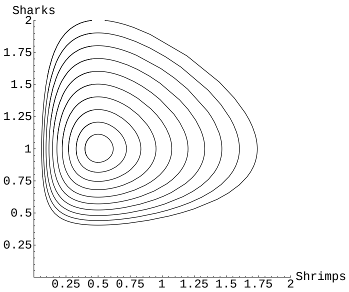
Plotting an orbit:

```
S[u_,v_]:=NDSolve[{x'[t]==x[t](6-2x[t]-y[t]),y'[t]==y[t](5-x[t]-y[t]),x[0]==u,y[0]==v},{x,y},{t,0,1}]
ParametricPlot[Evaluate[{x[t],y[t]}/.S[0.3,0.5]],{t,0,1},AspectRatio->1,AxesLabel->{"x[t]","y[t]"}]
```

VOLTERRA-LODKA SYSTEMS are systems of the form

$$\begin{aligned}\dot{x} &= 0.4x - 0.4xy \\ \dot{y} &= -0.1y + 0.2xy\end{aligned}$$

This example has equilibrium points $(0,0)$ and $(1/2,1)$.



It describes for example a shrimp-shark population. The shrimp population $x(t)$ becomes smaller when there are more sharks, the shark population grows with more shrimps. Volterra explained so first the oscillation of fish populations in the Mediterian sea.

EXAMPLE: HAMILTONIAN SYSTEMS are systems of the form

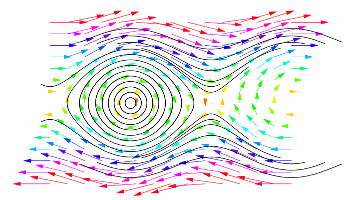
$$\begin{aligned}\dot{x} &= \partial_y H(x,y) \\ \dot{y} &= -\partial_x H(x,y)\end{aligned}$$

where H is called the **energy**. Usually, x is the position and y the momentum.

THE PENDULUM is an example, where $H(x,y) = y^2/2 - \cos(x)$ is the sum of the kinetic and potential energy.

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -\sin(x)\end{aligned}$$

x is the angle between the pendulum and y-axes, y the angular velocity.



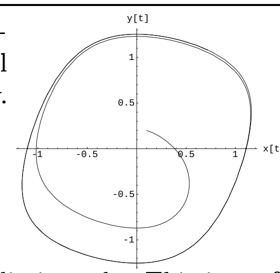
(More about the pendulum in the homework). Hamiltonian systems preserve energy $H(x,y)$ because $\frac{d}{dt}H(x(t),y(t)) = \partial_x H(x,y)\dot{x} + \partial_y H(x,y)\dot{y} = \partial_x H(x,y)\partial_y H(x,y) - \partial_y H(x,y)\partial_x H(x,y) = 0$. Orbits stay on level curves of H.

EXAMPLE: LIENHARD SYSTEMS are differential equations of the form $\ddot{x} + \dot{x}F'(x) + G'(x) = 0$. With $y = \dot{x} + F(x), G'(x) = g(x)$, this gives

$$\begin{aligned}\dot{x} &= y - F(x) \\ \dot{y} &= -g(x)\end{aligned}$$

VAN DER POL EQUATION $\ddot{x} + (x^2 - 1)\dot{x} + x = 0$ appears in electrical ingeneering, biology or biochemistry. There $F(x) = x^3/3 - x, g(x) = x$.

$$\begin{aligned}\dot{x} &= y - (x^3/3 - x) \\ \dot{y} &= -x\end{aligned}$$



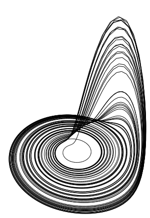
Lienhard systems have **limit cycles**. A trajectory always ends up on that limit cycle. This is useful for engineers, who need oscillators which are stable under changes of parameters. One knows: if $g(x) > 0$ for $x > 0$ and F has exactly three zeros $0, a, -a$, $F'(0) < 0$ and $F'(x) \geq 0$ for $x > a$ and $F(x) \to \infty$ for $x \to \infty$, then the corresponding Lienhard system has exactly one stable limit cycle.

CHAOS can occur for systems $\dot{x} = f(x)$ in three dimensions. Examples are 1D systems of the form $\ddot{x} = f(x,t)$ which becomes a three-dimensionional system in the coordinates $(x,y,z) = (x,\dot{x},t)$, systems $\ddot{x} = f(x,\dot{x})$ in two dimensions which becomes a four dimensional system in the coordinates $(x,\dot{x})$ or for systems $\dot{x} = f(x,t)$ in two dimensions. The term **chaos** has no uniform definition but usually means that one can find a true random number generator embedded in the system. Chaos theory is 100 years old. Basic insight had been obtained by Poincaré already. Since about 40 years, the subject exploded to a science, partly due to the availability of computers.

STRANGE ATRRACTORS IN 3D.

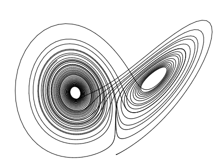
THE ROESSLER SYSTEM

$$\begin{aligned}\dot{x} &= -(y+z) \\ \dot{y} &= x + 0.2 * y \\ \dot{z} &= 0.2 + x * z - 5.7 * z\end{aligned}$$



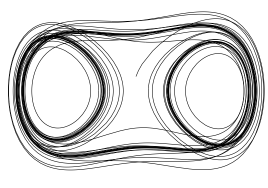
THE LORENTZ SYSTEM

$$\begin{aligned}\dot{x} &= 10(y-x) \\ \dot{y} &= -xz + 28x - y \\ \dot{z} &= xy - \frac{8z}{3}\end{aligned}$$



THE DUFFING SYSTEM $\ddot{x} + \dot{x}/10 - x + x^3 - 12\cos(t) = 0$ (metallic plate between magnets).

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -y/10 - x + x^3 - 12\cos(z) \\ \dot{z} &= 1\end{aligned}$$



Other chaotic examples can be obtained from mechanics like the **driven pendulum** $\ddot{x} + \sin(x) - \cos(t) = 0$.