Continuous Dynamical Systems II 4/18/2002 **Math 21b, O. Knill**

Homework for Tuesday April 23, 9.2: 8,14,22-26,34,35*,40*

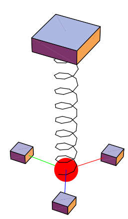
COMPLEX LINEAR 1D CASE. $\dot{x} = \lambda x$ for $\lambda = a + ib$ has solution $x(t) = e^{at}e^{ibt}x(0)$ and norm $||x(t)|| = e^{at}||x(0)||$.

OSCILLATOR: The system $\ddot{x} = -\lambda x$ has the solution $x(t) = \cos(\sqrt{\lambda}t)x(0) + \sin(\sqrt{\lambda}t)\dot{x}(0)/\sqrt{\lambda}$.

DERIVATION. $\dot{x} = y, \dot{y} = -\lambda x$ and in matrix form as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ \lambda & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$$

and because A has eigenvalues $\pm i\sqrt{\lambda}$, the new coordinates move as $a(t) = e^{i\sqrt{\lambda}t}a(0)$ and $b(t) = e^{-i\sqrt{\lambda}t}b(0)$. Writing this in the original coordinaes $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} a(t) \\ b(t) \end{bmatrix}$ and fixing the constants gives $x(t), y(t)$.



EXAMPLE. THE SPINNER. The spinner is a rigid body attached to a spring aligned around the z-axes. The body can rotate around the z-axes and bounce up and down. The two motions are coupled in the following way: when the spinner winds up in the same direction as the spring, the spring gets tightend and the body gets a lift. If the spinner winds up to the other direction, the spring becomes more relaxed and the body is lowered.

SETTING UP THE DIFFERENTIAL EQUATION.
x is the angle and y the height of the body. We put the coordinate system so that $y = 0$ is the point, where the body stays at rest if $x = 0$. We assume that if the spring is winded up with an angle x, this produces an upwards force x and a momentum force $-3 * x$. We furthermore assume that if the body is at position y, then this produces a momentum y onto the body and an upwards force y. The differential equations

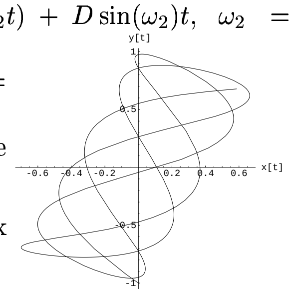
$$\begin{aligned} \ddot{x} &= -3x + y \\ \ddot{y} &= -y + x \end{aligned}$$

can be written as $\ddot{v} = Av = \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix} v$.

FINDING GOOD COORDINATES $w = S^{-1}v$ is obtained with getting the eigenvalues and eigenvectors of A: $\lambda_1 = -2 - \sqrt{2}$, $\lambda_2 = -2 + \sqrt{2}$ $v_1 = \begin{bmatrix} -1 - \sqrt{2} \\ 1 \end{bmatrix}$, $v_1 = \begin{bmatrix} -1 + \sqrt{2} \\ 1 \end{bmatrix}$ so that $S = \begin{bmatrix} -1 - \sqrt{2} & -1 + \sqrt{2} \\ 1 & 1 \end{bmatrix}$.

SOLVE THE SYSTEM $\ddot{a} = \lambda_1 a, \ddot{b} = \lambda_2 b$ IN THE GOOD COORDINATES $\begin{bmatrix} a \\ b \end{bmatrix} = S^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$.
$a(t) = A\cos(\omega_1 t) + B\sin(\omega_1)t$, $\omega_1 = \sqrt{-\lambda_1}$, $b(t) = C\cos(\omega_2 t) + D\sin(\omega_2)t$, $\omega_2 = \sqrt{-\lambda_2}$.

THE SOLUTION IN THE ORIGINAL COORDINATES. $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} a(t) \\ b(t) \end{bmatrix}$. At $t = 0$ we know $x(0), y(0), \dot{x}(0), \dot{y}(0)$. This fixes the constants in $x(t) = A_1\cos(\omega_1 t) + B_1\sin(\omega_1 t)$ $+A_2\cos(\omega_2 t) + B_2\sin(\omega_2 t)$. The curve $(x(t), y(t))$ traces a Lyssajoux figure:



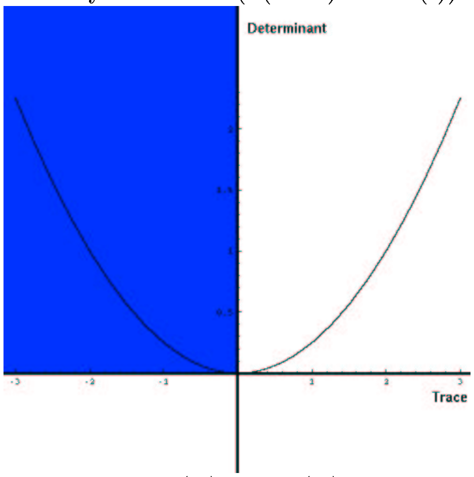
ASYMPTOTIC STABILITY.
A linear system $\dot{x} = Ax$ in the 2D plane is asymptotically stable if and only if $\det(A) > 0$ and $\mathrm{tr}(A) < 0$.

PROOF. If the eigenvalues λ_1, λ_2 of A are real then both beeing negative is equivalent with $\lambda_1\lambda_2 = \det(A) > 0$ and $\mathrm{tr}(A) = \lambda_1 + \lambda_2 < 0$. If $\lambda_1 = a + ib, \lambda_2 = a - ib$, then a negative a is equivalent to $\lambda_1 + \lambda_2 = 2a < 0$ and $\lambda_1\lambda_2 = a^2 + b^2 > 0$.

ASYMPTOTIC STABILITY COMPARISON OF DISCRETE AND CONTINUOUS SITUATION.
The trace and the determimant are independent of the basis, they can be computed fast, and are real if A is real. It is therefore convenient to determine the region in the tr − det-plane, where continuous or discrete dynamical systems are asymptotically stable. While the continuous dynamical system is related to a discrete system, it is important not to mix these two situations up.

Continuous dynamical system.

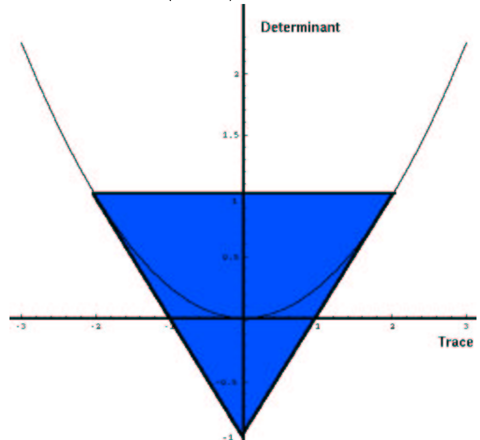
Stability of $\dot{x} = Ax$ ($x(t+1) = e^A x(t)$).



Stability in $\det(A) > 0, \mathrm{tr}(A) > 0$
Stability if $\mathrm{Re}(\lambda_1) < 0$, $\mathrm{Re}(\lambda_2) < 0$.

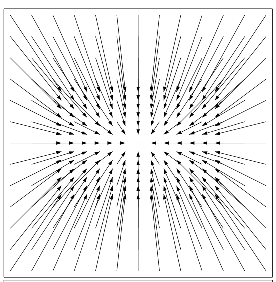
Discrete dynamical system.

Stability of $x(t+1) = Ax$

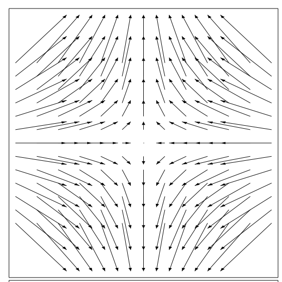


Stability in $|\mathrm{tr}(A)| - 1 < \det(A) < 1$
Stability if $|\lambda_1| < 1$,$|\lambda_2| < 1$.

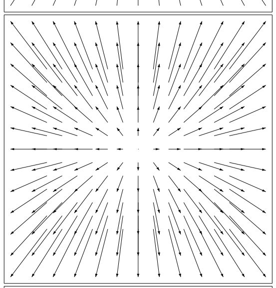
PHASEPORTRAITS. (In two dimensions we can plot the vector field, draw some trajectories)



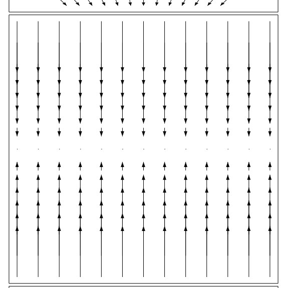
$\lambda_1 < 0, \lambda_2 < 0$,
i.e $A = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$



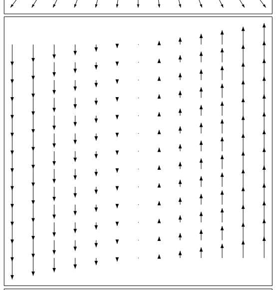
$\lambda_1 < 0, \lambda_2 > 0$,
i.e $A = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$



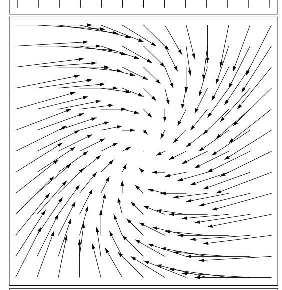
$\lambda_1 > 0, \lambda_2 > 0$,
i.e $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$



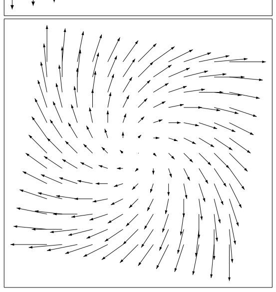
$\lambda_1 = 0, \lambda_2 < 0$,
i.e $A = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}$



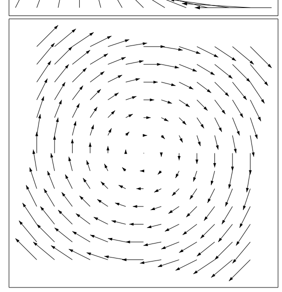
$\lambda_1 = 0, \lambda_2 = 0$,
i.e $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$



$\lambda_1 = a + ib, a < 0, \lambda_2 = a - ib$,
i.e $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$



$\lambda_1 = a + ib, a > 0, \lambda_2 = a - ib$,
i.e $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$



$\lambda_1 = ib, a < 0, \lambda_2 = -ib$,
i.e $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$