Orthogonality, Determinant, Eigensystems, 4/11/2002 **Math21b, O.Knill**

The EIGENVECTORS AND EIGENVALUES of a matrix A reveal the structure of A. Diagonalisation in general eases the computations with A. It allows to find explicit formulas for LINEAR DYNAMICAL SYSTEMS $x \mapsto Ax$. Such systems are important for example in probability theory. The dot product leads to the notion of ORTHOGONALITY, allows measurements of angles and lengths and leads to geometric notations like rotation, reflection or projection in arbitrary dimensions. Least square solutions of $Ax = b$ allow for example to solve fitting problems. DETERMINANTS of matrices appear in the definition of the characteristic polyomial and as volumes of parallelepipeds or as scaling values in change of variables. The notion allows to give explicit formulas for the inverse of a matrix or to solutions of $Ax = b$.

ORTHOGONAL $v \cdot w = 0$.
LENGTH $||v|| = \sqrt{v \cdot v}$.
UNIT VECTOR $||v|| = \sqrt{v \cdot v} = 1$.
ORTHOGONAL $v_1, \dots, v_n$: pairwise orthogonal.
ORTHONORMAL orthogonal and length 1.
ORTHONORMAL BASIS A basis which is orthonormal.
ORTHOGONAL TO V v is orthogonal to V if $v \cdot x = 0$ for all $x \in V$.
ORTHOGONAL COMPLEMENT OF V Linear space $V^\perp = \{v | v \text{ orthogonal to } V\}$.
PROJECTION ONTO V orth. basis $v_1, \dots, v_n$ in V, $\text{perp}_V(x) = (v_1 \cdot x)v_1 + \dots + (v_n \cdot x)v_n$.
GRAMM-SCHMIDT Recursive $u_i = v_i - \text{proj}_{V_{i-1}} v_i$, $w_i = u_i/||u_i||$ leads to orthonormal basis.
QR-FACTORIZATION $Q = [w_1 \cdots w_n]$, $R_{ii} = u_i$, $[R]_{ij} = w_i \cdot v_j, j > i$.
TRANSPOSE $[A^T]_{ij} = A_{ji}$. Transposition switches rows and columns.
SYMMETRIC $A^T = A$.
SKEWSYMMETRIC $A^T = -A$ ($\Rightarrow R = e^A$ orthogonal: $R^T = e^{A^T} = e^{-A} = R^{-1}$).
DOT PRODUCT AS MATRIX PRODUCT $v \cdot w = v^T \cdot w$.
ORTHOGONAL MATRIX $Q^T Q = 1$.
ORTHOGONAL PROJECTION onto V is AA^T, where v_i form orthonormal basis in V.
NORMAL EQUATION to $Ax = b$ is the consistent system $A^T Ax = A^T b$.
LEAST SQUARE SOLUTION of $Ax = b$ is $x^* = (A^T A)^{-1} A^T b$.
ANGLE between two vectors x, y is $\alpha = \arccos((x \cdot y)/(||x|| \, ||y||))$.
DETERMINANT $\det(A) = (\sum_{even\ \pi} - \sum_{odd\ \pi}) A_{1\pi(1)} A_{2\pi(2)} \cdots A_{n\pi(n)}$.
PARALLELEPIPED Image of unit cube by A. Spanned by columns of A, volume $\sqrt{\det(A^T A)}$.
MINOR A_{ij}, the matrix with row i and column j deleted.
CLASSICAL ADJOINT $\text{adj}(A)_{ij} = (-1)^{i+j} \det(A_{ji})$ (note switch of ij).
ORIENTATION $\text{sign}(\det(A))$ defines orientantation of column vectors of A.
TRACE is $\text{tr}(A) = \sum_i A_{ii}$, the sum of diagonal elements of A.
CHARACTERISTIC POLYNOMIAL $f_A(\lambda) = \det(\lambda - A) = \lambda^n - \text{tr}(A)\lambda + \dots + (-1)^n \det(A)$.
EIGENVALUES AND EIGENVECTORS $Av = \lambda v, v \neq 0$, eigenvalue λ, eigenvector v.
FACTORISATION OF $f_A(\lambda)$ Have formula $f_A(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$.
ALGEBRAIC MULTIPLICITY k If $f_A(\lambda) = (\lambda - \lambda_0)^k g(\lambda)$ with $g(\lambda_0) \neq 0$.
GEOMETRIC MULTIPLICITY The dimension of the kernel of $\lambda - A$.
KERNEL AND EIGENVECTORS Vectors in the kernel of A are eigenvectors of A.
EIGENBASIS Basis which consists of eigenvectors of A.
COMPLEX NUMBERS $z = x + iy = |z| \exp(i \arg(z)) = re^{i\phi} = r \exp(i\phi) = r\cos(\phi) + ir\sin(\phi)$.
MODULUS AND ARGUMENT $|z| = |x + iy| = \sqrt{x^2 + y^2}$, $\phi = \arg(z) = \arctan(y/x)$.
CONJUGATE $\overline{z} = x - iy$ if $z = x + iy$.
LOGARITHM $\log(z) = \log|z| + i \arg(z)$.
LINEAR DYNAMICAL SYSTEM Linear map $x \mapsto Ax$ defines orbit $x(t+1) = A(x(t))$.
ASYMPTOTIC STABILITY $A^n x \to 0$ for all x.

Skills checklist

COMPUTE DETERMIMANTS IN DIFFERENT WAYS (rref(A),Laplace,volume,patterns).
GRAMM-SCHMIDT ORTHOGONALISATION (algorithm).
COMPUTING EIGENVALUES OF A (factoring characteristic polynomial).
COMPUTING EIGENVECTORS OF A (determining kernel of $\lambda - A$).
COMPUTING ALGEBRAIC AND GEOMETRIC MULTIPLICITIES (know the definitions).
COMPUTING ORTHOGONAL PROJECTION ONTO LINEAR SUBSPACES (formula).
PRODUCE LEAST SQUARE SOLUTION OF LINEAR EQUATION (formula).
ALGEBRA OF MATRICES (multiplication, inverse, recall rank, image, kernel, inverse).
CALCULATION WITH COMPLEX NUMBERS (operations, identifying $\text{Re}(z), \text{Im}(z)$).
DIAGONALIZE MATRIX (find eigensystem, build S so that $S^{-1}AS$ is diagonal.).
DETERMINE STABILITY OF A LINEAR SYSTEM (eigenvalue criterion, especially 2D case).

ORTHOGONAL IMPLIES INDEPENDENT. Orthogonal vectors are linearly independent.
ORTHOGONAL PLUS SPAN IMPLIES BASIS. n orthogonal vectors $\mathbf{R}^n$ form a basis.
ORTHOGONAL COMPLEMENT IS LINEAR SPACE.
LINE TO PROJECTION IS ORTHOGONAL TO V. $x - \text{proj}_V(x)$ is orthogonal to V.
PYTHAGORAS: x, y orthogonal $\Rightarrow ||x+y||^2 = ||x||^2 + ||y||^2$.
PROJECTION CONTRACTS $||\text{proj}_V(x)|| \leq ||x||$.
IMAGE OF A IS ORTHOGONAL COMPLEMENT TO KERNEL OF A^T.
DIMENSION OF ORTHOGONAL COMPLEMENT. $\dim(V) + \dim(V^T) = n$.
CAUCHY-SCHWARTZ INEQUALITY: $|x \cdot y| \leq ||x|| \, ||y||$.
TRIANGLE INEQUALITY: $||x+y|| \leq ||x|| + ||y||$.
ROW VECTORS OF A are orthogonal to $\ker(A)$. Short $\text{im}(A^T) = \ker(A)^\perp$.
ORTHOGONAL TRANSFORMATIONS preserve angle, length. Columns form orthonormal basis.
ORTHOGONAL PROJECTION: $A(A^TA)^{-1}A^T$ onto V with $A = [v_1, \dots v_n], V = [v_1, \dots, v_n]$.
ORTHOGONAL PROJECTION: onto V is AA^T if $A = [v_1, \dots, v_n]$ is orthogonal.
ORTHOGONAL PROJECTIONS are not orthogonal transformations in general.
KERNEL OF A AND A^TA are the same: $\ker(A) = \ker(A^TA)$.
DETERMINANTS $\det(\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. $\det \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - fha - bdi$.
DETERMIMANT OF DIAGONAL OR TRIANGULAR MATRIX: product of diagonal entries.
DETERMINANT OF BLOCK MATRIX: $\det(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}) = \det(A)\det(B)$.
PROPERTIES OF DETERMINANTS. $\det(AB) = \det(A)\det(B)$, $\det(A^{-1}) = 1/\det(A)$.
PROPERTIES OF DETERMINANTS. $\det(SAS^{-1}) = \det(A)$, $\det(A^T) = \det(A)$.
DETERMINANT OF $[A]_{ij} = v_i \cdot v_j$ is square of determinant of $B = [v_1, \dots, v_n]$.
SWITCHING TWO ROWS. $\det(B) = -\det(A)$.
ADDING ROW TO GIVEN ROW: $\det(B) = \det(A)$
PARALLELEPIPED. $|\det(A)| = \text{vol}$(parallelepiped spanned by columns of A).
K-EPIPED. $\sqrt{\det(A^TA)} = \text{vol}$($k$-dimensional parallelepiped spanned by column vectors of A).
RREF. $\det(A) = (-1)^s (\prod_i \alpha_i) \det(\text{rref}(A))$ with α_i row scaling factors and s row switches.
IN ODD DIMENSIONS a real matrix has a real eigenvalue.
IN EVEN DIMENSIONS a real matrix with negative determinant has real eigenvalue.
PROPERTIES OF TRANSPOSE. $(A^T)^T = A$, $(AB)^T = B^TA^T$, $(A^{-1})^T = (A^T)^{-1}$.
DIAGONALISATION: A $n \times n$ matrix, S eigenvectors of A in columns, $S^{-1}AS$ diagonal.
NONTRIVIAL KERNEL $\Leftrightarrow \det(A) = 0$.
INVERTIBLE MATRIX $\Leftrightarrow \det(A) \neq 0$.
LAPLACE EXPANSION. $\det(A) = (-1)^{i+1} a_{i1} \det(A_{i1}) + \cdots + (-1)^{i+n} a_{in} \det(A_{in})$
ORTHOGONAL MATRICES A have $\det(A) = \pm 1$
ROTATIONS satisfy $\det(A) = 1$ in all dimensions.
ROTATIONS with angle ϕ in the plane have eigenvalues $\exp(i\phi)$.
QR DECOMPOSITION. $A = QR$ orthogonal A, upper triangular R. Have $|\det(A)| = \prod_{i=1}^n R_{ii}$.
CRAMER'S RULE. Solve $Ax = b$ by $x_i = \det(A_i)/\det(A)$, where A_i is A with b in column i.
CLASSICAL ADJOINT AND INVERSE. $A^{-1} = \text{adj}(A)/\det(A)$.
DETERMINANT IS PRODUCT OF EIGENVALUES. $\det(A) = \prod_i \lambda_i$.
TRACE IS SUM OF EIGENVALUES. $\text{tr}(A) = \sum_i \lambda_i$.
GEOMETRIC MULTIPLICITY $\leq$ ALGEBRAIC MULTIPLICITY.
DIFFERENT EIGENVALUES $\Rightarrow$ EIGENSYSTEM. $\lambda_i \neq \lambda_j, i \neq j \Rightarrow$ eigenvalues form basis.
EIGENVALUES OF A^T agree with eigenvalues of A (same char. polynomial).
RANK OF A^T is equal to the rank of A.
REFLECTION at linear k-dimensional subspace in $\mathbf{R^n}$ has determinant $(-1)^{(n-k)}$.
ARGUMENTS OF PRODUCTS ADD: $\arg(zw) = \arg(z) + \arg(w)$.
DE MOIVRE FORMULA: $z^n = \exp(in\phi) = \cos(n\phi) + i\sin(n\phi) = (\cos(\phi) + i\sin(\phi))^n$.
FUNDAMENTAL THEOREM OF ALGEBRA. $\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + \lambda_0$ has n roots.
NUMBER OF EIGENVALUES. A $n \times n$ matrix has exactly n eigenvalues (count multiplicity).
POWER OF A MATRIX. A^n has eigenvalues λ^n if A has eigenvalue λ.
EIGENVALUES OF $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are $\lambda_\pm = \text{tr}(A)/2 \pm \sqrt{(\text{tr}(A)/2)^2 - \det(A)}$.
EIGENVECTORS OF $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $c \neq 0$ are $v_\pm = [\ \lambda_\pm - d, c\ ]$.
CRITERION FOR LINEAR STABILITY. Asymptotic stability $\Leftrightarrow |\lambda_i| < 1$ for all i.
ROTATION-DILATION MATRIX: $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$, eigenvalues $p \pm iq$, eigenvectors $(\pm i, 1)$.
ROTATION-DILATION MATRIX: linear stable origin if and only if $|\det(A)| < 1$.
STABILITY TRIANGLE: 2×2 case: asymptotic stability in triangle $|\text{tr}(A)| - 1 < \det(A) < 1$.