

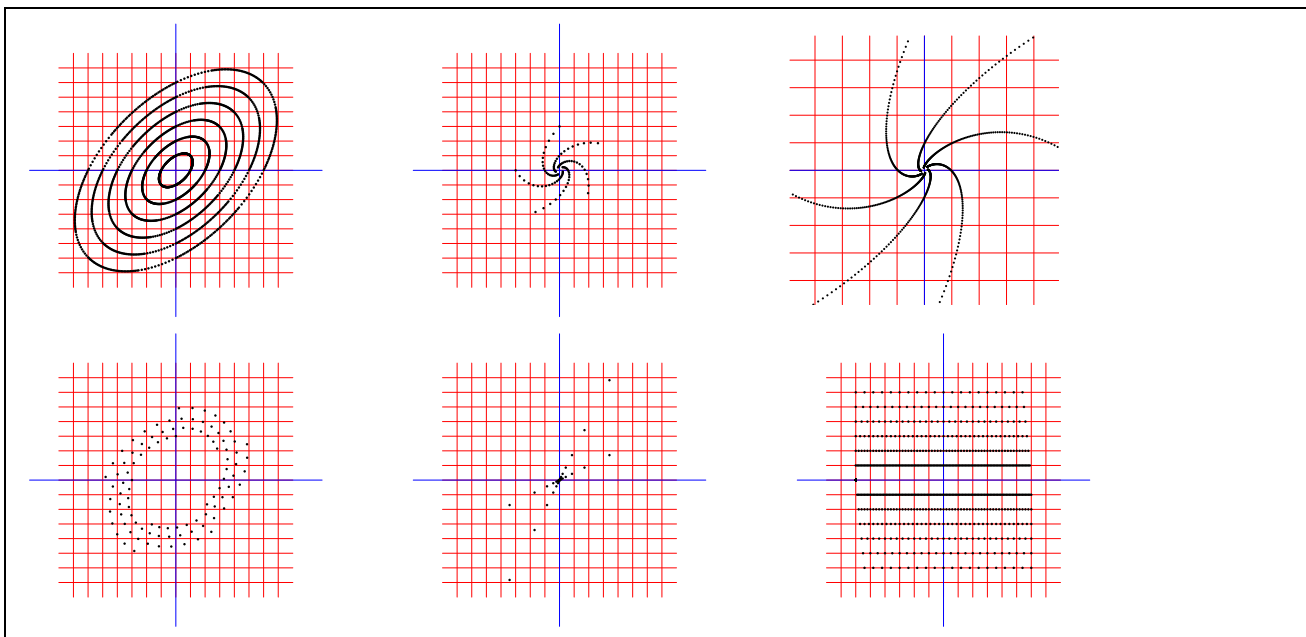
Homework for Thursday, April 11, 2002: Section 7.6, 8,12,22,28,40afg,42*

LINEAR DYNAMICAL SYSTEM. A linear map $x \mapsto Ax$ defines a **dynamical system**. Iterating the map produces an **orbit** $x_0, x_1 = Ax, x_2 = A^2 = AAx, \dots$. The vector $x_n = A^n x_0$ describes the situation of the system at **time** n .

QUESTIONS. We are interested in questions like:

Where does x_n go when time evolves? Can one describe what happens asymptotically when time n goes to infinity?

In the case of the Fibonacci sequence x_n which gives the number of rabbits in a rabbit population at time n , the population grows essentially exponentially. Linear algebra (see the homework) gave us the rate as the golden mean: $x_{n+1}/x_n = (\sqrt{5} + 1)/2$. Such a behavior would be called **unstable**.



ASYMPTOTIC STABILITY. The origin 0 is invariant under a linear map $A : x \mapsto Ax$. It is called **asymptotically stable** if $A^n(x) \rightarrow 0$ for all $x \in \mathbb{R}^n$.

EXAMPLE 1. Remember the chemical spill into the Swiss lakes? There all $A^n(x) \rightarrow 0$, where x is a three dimensional vector giving the amount of chemicals in each of the three lakes.

EXAMPLE 2. Let $A = \begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ be a dilation rotation matrix. Multiplication with such a matrix is analogue to the multiplication of complex numbers $p + iq$. So, A^n is the dilation rotation matrix corresponding to $(p + iq)^n$. Now, $|(p + iq)^n| = |p + iq|^n$ and the origin is asymptotically stable if and only if $|p + iq| < 1$. Because $\det(A) = |p + iq|^2$, rotation-dilation matrices A have an asymptotically stable origin if and only if $|\det(A)| < 1$.

EXAMPLE 3. If a matrix A has an eigenvalue $|\lambda| \geq 1$ to the eigenvector v , then $A^n v = \lambda^n v$ whose length is $|\lambda|^n$ times the length of v . So, we have no asymptotic stability if an eigenvalue satisfies $|\lambda| \geq 1$.

STABILITY. The book also writes "stable" for "asymptotically stable". This is ok to abbreviate. Note however that commonly the term "stable" also includes rotations or the identity. We would prefer to leave always the attribute "asymptotic".

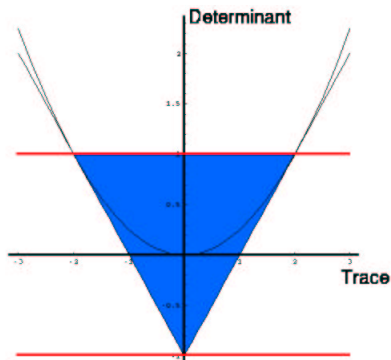
DILATION-ROTATION MATRICES. Dilation-rotation matrices $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ have the eigenvalues $p \pm iq$.

ROTATIONS. Rotations $\begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$ have the eigenvalue $\exp(\pm i\phi) = \cos(\phi) + i \sin(\phi)$ and are not asymptotically stable. **DILATIONS.** Dilations $\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$ have the eigenvalue r with algebraic and geometric multiplicity 2. Dilations are asymptotically stable if $|r| < 1$.

CRITERION. A linear dynamical system $x \mapsto Ax$ has an asymptotically stable origin if and only if all eigenvalues have absolute value < 1 .

PROOF. We have already seen in Example 3, that if one eigenvalue satisfies $|\lambda| > 1$, then the origin is not asymptotically stable. Assume now that $|\lambda_i| < 1$ for all i . Assumeing also that all eigenvalues are different, we have an eigenbasis v_1, \dots, v_n . Every x can be written as $x = \sum_{j=1}^n x_j v_j$. Then, $A^n x = A^n (\sum_{j=1}^n x_j v_j) = \sum_{j=1}^n x_j \lambda_j^n v_j$ and because $|\lambda_j|^n \rightarrow 0$, there is stability. The proof of the general case will be accessible later.

THE 2-DIMENSIONAL CASE. The characteristic polynomial of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $f_A(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \det(A)$. The eigenvalues are $\lambda_{\pm} = \text{tr}(A)/2 \pm \sqrt{(\text{tr}(A)/2)^2 - \det(A)}$, the eigenvectors are $v_{\pm} = [\lambda_{\pm} - d, c]$. If the discriminant $(\text{tr}(A)/2)^2 - \det(A)$ is nonnegative, then the eigenvalues are real. This happens below the parabola, where the discriminant is zero.



CRITERION. A linear dynamical system in two dimensions is asymptotically stable if and only if $(\text{tr}(A), \det(A))$ is inside the **stability triangle** bounded by the lines $\det(A) = 1, \det(A) = \text{tr}(A) - 1$ and $\det(A) = -\text{tr}(A) - 1$.

PROOF. Write $T = \text{tr}(A)/2, D = \det(A)$. If $|D| \geq 1$, there is no asymptotic stability. If $\lambda = T + \sqrt{T^2 - D} = \pm 1$, then $T^2 - D = (\pm 1 - T)^2$ and $D = 1 \pm 2T$. For $D \leq -1 + |2T|$ we have a real eigenvalue ≥ 1 . The conditions for stability is therefore $D > |2T| - 1$. It implies automatically $D > -1$ so that the triangle can be described shortly as $|\text{tr}(A)| - 1 < \det(A) < 1$.

EXAMPLES.

- 1) The matrix $A = \begin{bmatrix} 1 & 1/2 \\ -1/2 & 1 \end{bmatrix}$ has the determinant $5/4$ and trace 2 and the origin is unstable. It is a dilation-rotation matrix which corresponds to the complex number $1 + i/2$ which has an absolute value > 1 .
- 2) A rotation A is never asymptotically stable: $\det(A) = 1$ and $\text{tr}(A) = 2 \cos(\phi)$. Rotations are the upper side of the **stability triangle**.
- 3) A dilation is asymptotically stable if and only if the scaling factor has norm < 1 .
- 4) If $\det(A) = 1$ and $\text{tr}(A) < 2$ then the eigenvalues are on the unit circle and there is no asymptotic stability.
- 5) If $\det(A) = -1$ (example Fibonacci) there is no asymptotic stability. For $\text{tr}(A) = 0$, we are at a corner of the stability triangle and the map is a reflection.

SOME PROBLEMS.

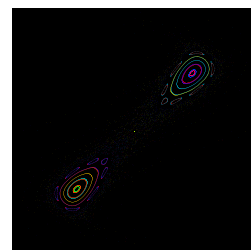
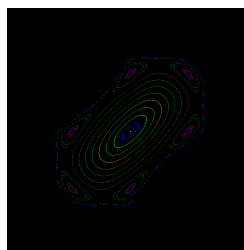
- 1) If A is a matrix with asymptotically stable origin, what is the stability of 0 with respect to A^T ?
- 2) If A is a matrix which has an asymptotically stable origin, what is the stability with respect to A^{-1} ?
- 3) If A is a matrix which has an asymptotically stable origin, what is the stability with respect to A^{100} ?

ON THE STABILITY QUESTION.

For general dynamical systems, the question of stability can be very difficult. We deal here only with linear dynamical systems, where the eigenvalues determine everything. For nonlinear systems, the story is not so simple even for simple maps like the Henon map (see below). The questions go deeper: it is for example not known, whether our solar system is stable. We don't know whether in some future, one of the planets could get expelled from the solar system (this is a mathematical question because the escape time would be larger than the life time of the sun). For other dynamical systems like the atmosphere of the earth or the stock market, we would really like to know what happens in the near future ...



A pioneer in stability theory was Aleksandr Lyapunov (1857-1918). For nonlinear systems like $x_{n+1} = gx_n - x_n^3 - x_{n-1}$ the stability of the origin is nontrivial. As with Fibonacci, this can be written as $(x_{n+1}, x_n) = (gx_n - x_n^2 - x_{n-1}, x_n) = A(x_n, x_{n-1})$ called **cubic Henon map** in the plane. To the right are orbits in the cases $g = 1.5, g = 2.5$.



The first case is stable (but proving this requires a theory called KAM), the second case is unstable (in this case actually the linearization at 0 determines the picture).