CALCULATING EIGENVECTORS 4/4/2002 — Math 21b, O.Knill

Homework for Tuesday, April 9, Section 7.3 problems 14,20,32,38*,42*,48.
Section 7.4 problems on Thursday

NOTATION. We often just write 1 instead of the identity matrix 1_n.

COMPUTING EIGENVALUES. Recall: because $\lambda - A$ has $\vec{v}$ in the kernel if λ is an eigenvalue the characteristic polynomial $f_A(\lambda) = \det(\lambda - A) = 0$ has eigenvalues as roots.

2×2 CASE. The characteristic polynomial of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $f_A(\lambda) = \lambda^2 - (a+d)/2\lambda + (ad - bc)$. The eigenvalues are $\lambda_\pm = T/2 \pm \sqrt{(T/2)^2 - D}$, where $T = a + d$ is the trace and $D = ad - bc$ is the determinant of A. If $(T/2)^2 \geq D$, then the eigenvalues are real. Away from that parabola in the (T, D) space, there are two different eigenvalues. The map A contracts volume for $|D| < 1$.

NUMBER OF ROOTS. There are examples with no real eigenvalue (i.e. rotations). By inspecting the graphs of the polynomials, one can deduce that $n \times n$ matrices with odd n always have a real eigenvalue. Also $n \times n$ matrixes with even n and a negative determinant always have a real eigenvalue.

HOW TO COMPUTE EIGENVECTORS? Because $(\lambda - A)\vec{v} = 0$, the vector $\vec{v}$ is in the kernel of $\lambda - A$.

EIGENVECTORS of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $c \neq 0$ to eigenvalues $\lambda_\pm = \mathrm{tr}(A)/2 \pm \sqrt{(\mathrm{tr}(A)/2)^2 - \det(A)}$ are $\vec{v}_\pm = \begin{bmatrix} \lambda_\pm - d, c \end{bmatrix}$.

ALGEBRAIC MULTIPLICITY. If $f_A(\lambda) = (\lambda - \lambda_0)^k g(\lambda)$, where $g(\lambda_0) \neq 0$, then f has **algebraic multiplicity** k. The algebraic multiplicity counts the number of times, an eigenvector occurs.

EXAMPLE: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ has the eigenvalue $\lambda = 1$ with algebraic multiplicity 2.

GEOMETRIC MULTIPLICITY. The dimension of the eigenspace E_λ of an eigenvalue λ is called the **geometric** multiplicity of λ.

EXAMPLE 1: The matrix of a shear is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. It has the eigenvalue 1 with algebraic multiplicity 2. The kernel of $A - 1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the geometric multiplicity is 1. It is different from the algebraic multiplicity.

EXAMPLE 2: (See example 3 in the book). The matrix $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ has the eigenvalue 1 with algebraic multiplicity 2 and the eigenvalue 0 with multiplicity 1. Eigenvectors to the eigenvalue $\lambda = 1$ are in the kernel of $A - 1 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. The kernel is spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and the geometric multiplicity is 1.

RELATION BETWEEN ALGEBRAIC AND GEOMETRIC MULTIPLICIY. (Proof later in the course). The geometric multiplicity is smaller or equal than the algebraic multiplicity.

PRO MEMORIA. You can remember this when you recall that the **geometric mean** $\sqrt{ab}$ of two numbers is smaller or equal to the **algebraic mean** $(a + b)/2$ of the two numbers. (This fact is totally* unrelated to the above fact and a mere coincidence of expressions, but it helps to remember it). Quite deeply burried there is a connection in terms of convexity. But this is rather philosophical. .

CASE WHEN ALL EIGENVALUES ARE DIFFERENT. If all eigenvalues are different, then the eigenvectors are all linearly independent. (The geometric and algebraic multiplicities are then all 1).
PROOF. Let λ_i be an eigenvalue different from 0 and assume the eigenvectors are linearly dependent. We can then write $v_i = \sum_{j \neq i} a_j v_j$. Now $\lambda_i v_i = A v_i = A(\sum_{j \neq i} a_j v_j) = \sum_{j \neq i} a_j \lambda_j v_j$ so that $v_i = \sum_{j \neq i} b_j v_j$ with $b_j = a_j \lambda_j / \lambda_i$. If the eigenvalues are different, then $a_j \neq b_j$ and by subtracting we get $0 = \sum_{j \neq i} (b_j - a_j) v_j = 0$. We see that $n-1$ eigenvectors of the n eigenvectors are linear dependent. You can from this build an induction argument.

CONSEQUENCE. If all eigenvalues of a $n \times n$ matrix A are different, there is an **eigenbasis**, a basis consisting of eigenvectors.

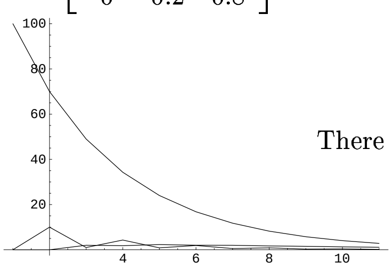
EXAMPLE. $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ has eigenvalues 1, 3 to the eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

EXAMPLE. (Problem 28 in the book). The upper triangular matrix J_n has λ in the diagonal and 1 directly above the diagonal. Find the algebraic and geometric multiplicities of the eigenvalues of J_n.
SOLUTION. The algebraic multiplicity of the eigenvalue λ is n. To get the kernel of $J_n - \lambda$ one solves the system of equations $x_{n-1} = 0, x_{n-2} = 0, \dots, x_1 = 0$. The geometric multiplicity is 1.

EXAMPLE. (Problem 40 in the text).

Photos of the Swiss lakes in the text. The pollution story is fiction fortunately.

The vector $A^n(x)b$ givs the pollution levels in the three lakes (Silvaplana, Sils, St Moritz) after n weeks, where $A = \begin{bmatrix} 0.7 & 0 & 0 \\ 0.1 & 0.6 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix}$ and $b = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$ is the initial pollution.



There is an eigenvector $e_3 = v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ to the eigenvalue $\lambda_3 = 0.8$. (What does the equation $Av_3 = 0.8 v_3$ mean?) There is an eigenvector $v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ to the eigenvalue $\lambda_2 = 0.6$. There is further an eigenvector $v_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ to the eigenvalue $\lambda_1 = 0.7$. We know $A^n v_1, A^n v_2$ and $A^n v_3$ explicitely.
How do we get the explicit solution $A^n b$? Because $b = 100 \cdot e_1 = 100(v_1 - v_2 + 3v_3)$, we have

$$\begin{aligned} A^n(b) &= 100 A^n (v_1 - v_2 + 3v_3) = 100(\lambda_1^n v_1 - \lambda_2^n v_2 + 3\lambda_3^n v_3) \\ &= 100 \left(0.7^n \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + 0.6^n \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + 3 \cdot 0.8^n \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 100(0.7)^n \\ 100(0.7^n + 0.6^n) \\ 100(-2 \cdot 0.7^n - 0.6^n + 3 \cdot 0.8^n) \end{bmatrix} \end{aligned}$$