FINDING EIGENVALUES 4/2/2002 **Math 21b, O.Knill**

Homework for Thursday, April 4, Section 7.2, 8,10,16,22,28*,32

EIGENVALUES AND EIGENVECTORS. Recall: a nonzero vector v is called an **eigenvector** of A if $Av = \lambda v$ for some number λ which is called an **eigenvalue**. How do we compute eigenvectors and eigenvalues?

THE TRACE. The **trace** of a matrix A is the sum of its diagonal elements.

CHARACERISTITIC POLYNOMIAL. If λ is an eigenvalue of A with eigenfunction v, then $A - \lambda$ has v in the kernel. Especially, $A - \lambda$ is not invertible and therefore $p(\lambda) = \det(\lambda - A) = 0$.
The function $\lambda \mapsto p(\lambda)$ is a polynomial in λ of the form $\lambda^n - \text{tr}(A)\lambda^{n-1} + \cdots + (-1)^n \det(A)$. It is called the **characteristic polynomial of** A. The eigenvalues are the roots of this polynomial.

THE 2x2 CASE. The characteristic polynomial of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $f_A(\lambda) = \lambda^2 - (a+d)/2\lambda + (ad - bc)$. The eigenvalues are $\lambda_\pm = T/2 \pm \sqrt{(T/2)^2 - D}$, where T is the trace and D is the determinant. In order that this is real, we must have $(T/2)^2 \geq D$. Away from that parabola there are two different eigevalues. The map A contracts volume for $|D| < 1$.
EXAMPLE. The characteristic polynomial of $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ is $\lambda^2 - \lambda - 1$.

BACK TO THE FIBONNACCI RABBITS. Fibonacci's recursion $u_{n+1} = u_n + u_{n-1}$ gives the size of a rabbit population in the year n. How do we get a solution? One approach is to postulate $u_n = \lambda^n$. If one plugs that into the equation, we get $\lambda^{n+1} = \lambda^n + \lambda^{n-1}$ or $\lambda^2 = \lambda + 1$. This has the solutions $\lambda_\pm = (1 \pm \sqrt{5})/2$ so that $u_n = a\lambda_1^n + b\lambda_2^n$, where a and b are arbitrary constants. In order that $u_0 = 0$ and $u_1 = 1$, the constants satisfy $a + b = 0, a\lambda_1 + b\lambda_2 = 0$. This gives $a = 1/\sqrt{5}, b = -1/\sqrt{5}$ and so $u_n = 5^{-1/2}\left[(1+\sqrt{5})^n/2^n - (1-\sqrt{5})^n/2^n\right]$.

Actually, the numbers λ_i are the eigenvalues of the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ which describes the recursion $\begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix} = A \begin{bmatrix} u_n \\ u_{n-1} \end{bmatrix}$. For an eigenvector $\vec{v}_i$ of A, we have $A^n \vec{v}_i = \lambda_i^n \vec{v}_i$.

ALGEBRAIC MULTIPLICITY. If $f_A(\lambda) = (\lambda - \lambda_0)^k g(\lambda)$, where $g(\lambda_0) \neq 0$, then f has **algebraic multiplicity** k.

EXAMPLE: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ has the eigenvalue $\lambda = 1$ with algebraic multiplicity 2.

HOW TO COMPUTE EIGENVECTORS? Because $(\lambda - A)v = 0$, the vector v is in the kernel of $\lambda - A$. We know how to compute this.

EXAMPLE: Back to Fibonacci. The kernel of $\begin{bmatrix} \lambda_\pm - 1 & -1 \\ -1 & \lambda_\pm \end{bmatrix}$ is $v_\pm = (1 \pm \sqrt{5})/2, 1)$. Because $(0,1) = (v_+ - v_-)/\sqrt{5}$ and $A^n v_+ = \lambda_+^n v_+, A^n v_- = \lambda_-^n v_-$, we get $(u_n, u_{n-1}) = A^n(0,1)$ with $u_n = (\lambda_+^n - \lambda_-^n)/\sqrt{5}$.

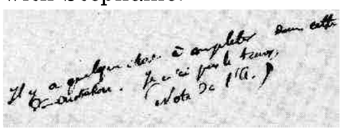
ROOTS OF POLYNOMIALS.
For polynomials of degree 3 and 4 explicit formulas in terms of radicals exist. As Galois (1811-1832) and Abel (1802-1829) have shown it is not possible in general for equations of degree 5 or higher.

REAL SOLUTIONS. A $(2n+1) \times (2n+1)$ matrix A always has a real eigenvalue because the characteristic polynomial $p(x) = x^5 + ... + \det(A)$ has the property that $p(x)$ goes to $\pm\infty$ for $x \to \pm\infty$. There exist therefore values a, b such that $p(a) < 0$ and $p(b) > 0$. By the intermediate value theorem, there exists a real x with $p(x) = 0$.

Application: every rotation A in 11 dimensional space has a direction v such that $A\vec{v} = \pm\vec{v}$. Reason: an orthogonal matrix can not have eigenvalues λ with $|\lambda| \neq 1$ because orthogonal transformations preserve the length of vectors.

GALOIS ... Galois raised his glass. With an open dagger in his hand he made threats against the King, Louis-Phillipe. Galois got arrested and emprisoned. At his trial his defence lawyer claimed that Galois had said "To Louis-Phillipe, if he betrays" but the last words had been drowned by the noise. Galois was acquitted. The 14th July was Bastille Day and Galois got arrested again because he was illigally swearing the uniform of the Artillery of the National Guard. He was also carrying a loaded rifle, several pistols and a dagger. Galois was sent back to Sainte-Plagie prison.

While in Sainte-Plagie prison Galois attempted to commit suicide by stabbing himself with a dagger but the other prisoners prevented him. While drunk in prison he poured out his soul "Do you know what I lack my friend? I confide it only to you: it is someone whom I can love and love only in spirit. I have lost my father and no one has ever replaced him, do you hear me...? " In March 1832 a cholera epidemic swept Paris and prisoners, including Galois, were transferred to the pension Sieur Faultrier. There he apparently fell in love with Stephanie-Felice du Motel, the daughter of the resident physician. After he was released on 29 April Galois exchanged letters with Stephanie whose name appears several times as a marginal note in one of Galois' manuscripts. It is not clear why Galois fought a duel with Perscheux d'Herbinville but it was certainly linked with Stephanie.



The note at the margin of the manuscript that Galois wrote the night before the duel. The note reads: "There is something to complete in this demonstration. I do not have the time. (Author's note)"

It is this which has led to the legend that he spent his last night writing out all he knew about group theory. Galois was wounded in the duel and was abandoned by d'Herbinville and his own seconds and found by a peasant. He died in Cochin hospital on 31 May and his funeral was held on 2 June. It was the focus for a Republican rally and riots followed which lasted for several days. Galois' brother and his friend Chevalier copied his mathematical papers and sent them to Gauss, Jacobi and others. It had been Galois' wish that Jacobi and Gauss should give their opinions on his work. No record exists of any comment these men made. However the papers reached Liouville who, in September 1843, announced to the Academy that he had found in Galois' papers a concise solution ...as correct as it is deep of this lovely problem: Given an irreducible equation of prime degree, decide whether or not it is soluble by radicals. Liouville published these papers of Galois in his Journal in 1846. The theory that Galois outlined in these papers is now called **Galois theory**. ....

ABEL. .... In 1815 Abel and his older brother were sent to the Cathedral School in Christiania. The founding of the University of Christiania had taken away the good teachers from the Cathedral School to staff the University when it opened for teaching in 1813. What had been a good school was in a bad state when Abel arrived. Uninspired by the poor school, he proved a rather ordinary pupil with some talent for mathematics and physics.

When a new mathematics teacher Bernt Holmboe joined the school in 1817 things changed markedly for Abel. The previous mathematics teacher had been dismissed for punishing a boy so severely that he had died. Abel began to study university level mathematics texts and, within a year of Holmboe's arrival, Abel was reading the works of Euler, Newton, Lalande and d'Alembert. Holmboe was convinced that Abel had great talent and encouraged him greatly taking him on to study the works of Lagrange and Laplace. However, in 1820 tragedy struck Abel's family when his father died.

Abel's father had ended his political career in disgrace by making false charges against his colleagues in the Storting after he was elected to the body again in 1818. His habits of drinking to excess also contributed to his dismissal and the family was therefore in the deepest trouble when he died. There was now no money to allow Abel to complete his school education, nor money to allow him to study at university and, in addition, Abel had the responsibility of supporting his mother and family. ...

(Pictures and (partly adapted) text from http://turnbull.dcs.st-and.ac.uk/history. Turn to that website to read more).