EIGENVALUES & DISCRETE DYNAMICAL SYSTEMS 3/21/2002 Math 21b, O.Knill

Homework for Tuesday, April 2, Section 7.1, 2,10,20,28,38*,40

EIGENVALUES AND EIGENVECTORS.
A nonzero vector v is called an **eigenvector** of A if $Av = \lambda v$ for some number λ which is called an **eigenvalue**.

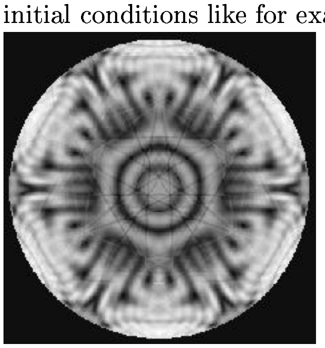
EXAMPLES.

- v is an eigenvector to the eigenvalue 0 if and only if it is in the kernel of A.
- If v is an eigenvector to the eigenvalue 1, then $Av = v$. Example: a vector in the axis of rotation of a rotation A.
- If A is a diagonal matrix with diagonal elements a_i, then the basis vectors e_i are eigenvectors.
- A shear A in the direction v has an eigenvector v.
- A rotation in the plane by an angle 30 degrees has no eigenvector. (There are actually eigenvectors but they are complex).

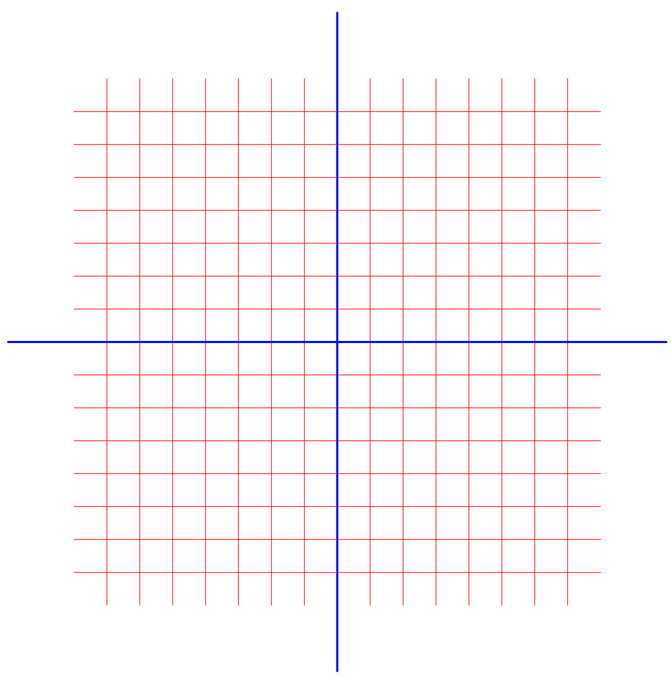
LINEAR DYNAMICAL SYSTEMS.
Iterating a linear map $x \mapsto Ax$ appears in many applications. One wants to understand what happens with $x_1 = Ax, x_2 = AAx = A^2x, x_3 = AAAx = A^3x, ...$

One dimension: $x \mapsto ax$ or $x_{n+1} = ax_n$ has the solution $x_n = a^n x_0$. For example, $1.03^{20} \cdot 1000 = 1806.11$ is the balance on a bank account which had 1000 dollars 20 years ago and if the interest rate was constant 3 percent. In many cases the behavior of u_{n+1} does not only depend on u_n but also on u_{n-1} or earlier times. In that case we write $(x_n, y_n) = (u_n, u_{n-1})$ and get a linear map. x_{n+1}, y_{n+1} depend in a linear way on x_n, y_n. We see two examples below.

EXAMPLE. A linear recursion problem: (motivated from quantum mechanics) How does u_n grow if $u_{n+1} + u_{n-1} = u_n$ and $u_0 = 0, u_1 = 1$. Plot $(x_n, y_n) = (u_n, u_{n-1})$ for different initial conditions like for example $(2, 0)$,$(0, 4)$.



(The picture shows electron diffraction patterns calculated using **Bloch waves**. By the way, the u_n just calculated is an example of a Bloch wave in a one dimensional crystal). Picture Source: P. Stadelman, 1995, *cimesg1.epfl.ch/CIOL/asu94/ICT_5.html*)

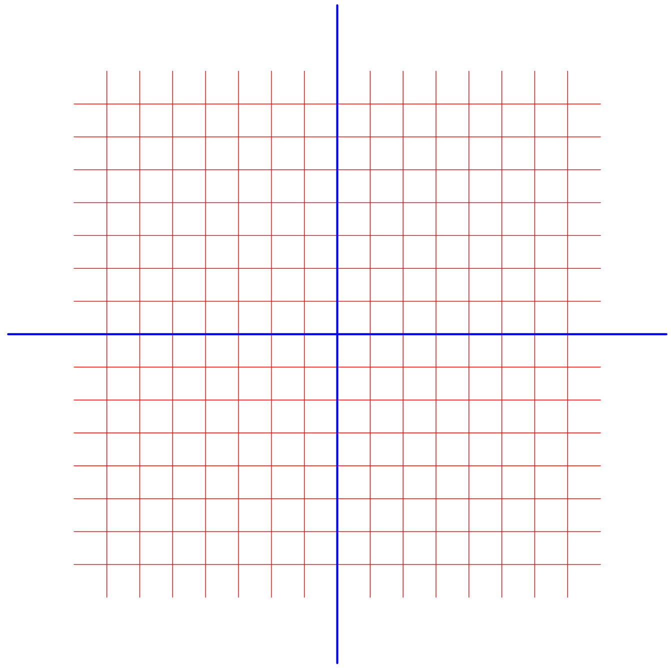


LINEAR RECURSION PROBLEM:

A problem in the third section of Liber abbaci, published in 1202 by **Leonardo Fibonacci** (1170-1250):

> *A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?*

How does u_n grow, if $u_{n+1} = u_n + u_{n-1}$? Plot $(x_n, y_n) = (u_n, u_{n-1})$ for different initial conditions like $(1,0)$ or $(0,1)$ and also $n < 0$.



WHERE DO LINEAR DYNAMICAL SYSTEMS APPEAR?

Linear systems $x \mapsto Ax$ appear in many places, like quantum mechanics, chaos theory, probability theory economics or biology. More examples are still to come.

EXAMPLE 1: Quantum mechanics. Some quantum mechanical systems of a particle in a potential V are described by $(Lu)_n = u_{n+1} + u_{n-1} + V_n u_n$. Energies E for which $(Lu)_n = Eu_n$, we have the recursion $u_{n+1} + u_{n-1} = (E - V_n)u_n$, when the potential is periodic in n, then this leads to a linear recursion problem. For example, if $V_n = V$ is constant, then $u_{n+1} + u_{n-1} = (E - V)u_n$. A question is for which E the solutions stay bounded. You have seen above the case $E - V = 1$.

EXAMPLE 2: Chaos theory. In plasma physics, one studies maps like $(x,y) \mapsto (2x - y - a\sin(x), x)$. You see that $(0,0)$ is a fixed point. Near that fixed point, the map is described by its Jacobian $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2-a & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. For which a is this linear system stable near $(0,0)$ in the sense that a point near $(0,0)$ stays nearby? The answer will be given using eigenvalues.

EXAMPLE 3: Evolution of quantities. Example could be market systems, population quantities of different species, or ingredient quantities in a chemical reaction. A linear description might not always be be a good model but it has the advantage that we can solve the system explicitly. Eigenvectors will provide the key to do so.

EXAMPLE 4: Markov Processes. The percentage of people using Apple OS or the Gnu/linux operating system is represented by a vector $\begin{bmatrix} m \\ l \end{bmatrix}$. Let $2/3$ be the percentage of Mac OS users, who switch to Linux each month and $1/2$ the percentage of Linux OS users, who switch to Apple. We look at the matrix $P = \begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix}$. (It is called a **Markov matrix**: the entries satisfy $0 \leq P_{ij} \leq 1$ and the sum of each column elements is equal to 1). What ratio of Apple/Linux users do we have? We can simulate this as follows with a dice: start in the state M =(1,0) (Mac). In the state M, if 3,4,5 or 6 shows up, switch to L=(0,1) otherwise keep M. If in the state L, if 1,2 or 3 shows up, switch to M otherwise keep L. The matrix P has an eigenvector $(3/7, 4/7)$ to the eigenvalue 1. The interpretation is that eventually, the probability to be in state M is $3/7$ and the probability to be in state L is $4/7$.