Determinant Properties 3/19/2002 **Math 21b, O. Knill**

Homework for Thursday, March 21, 6.2 Nr. 3,4,6,16*,36,40*

REMINDER. The **determinant** of a **square matrix** $A = a_{ij}$ was defined as the sum over all possible products $(-1)^\pi a_{1\pi(1)} \cdots a_{n\pi(n)}$, where $(-1)^\pi$ is the sign of the permutation (pattern).

TWO CASES.

The determinant of a **diagonal** or **tridiagonal** matrix is the product of its diagonal elements.

Example: $\det(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 2 & 3 & 4 & 0 \\ 1 & 1 & 2 & 1 \end{bmatrix}) = 20$.

The determinant of a partitioned matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is the product $\det(A)\det(B)$.

Example $\det(\begin{bmatrix} 3 & 4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 2 & 2 \end{bmatrix}) = 2 * 12 = 24$.

LINEARITY OF THE DETERMINANT. If the columns of A and B are the same except for the i'th column, then

$$\det([v_1, ..., v, ...v_n]) + \det([v_1, ..., w, ...v_n]) = \det([v_1, ..., v + w, ...v_n]) \ .$$

In general:

$$\det([v_1, ..., kv, ...v_n]) = k \ \det([v_1, ..., v, ...v_n]) \ .$$

The same holds for rows.

PROPERTIES OF DETERMINANTS.

$$\det(AB) = \det(A)\det(B) \ . \qquad \det(SAS^{-1}) = \det(A) \ . \qquad \det(\lambda A) = \lambda^n \det(A)$$

$$\det(A^{-1}) = \det(A)^{-1} \qquad \det(A^T) = \det(A) \qquad \det(-A) = (-1)^n \det(A) \ .$$

If B is obtained from A by switching two rows, then $\det(B) = -\det(A)$. If B is obtained by adding an other row to a given row, then this does not change the value of the determinant.

PROOF OF $\det(AB) = \det(A)\det(B)$, one brings the $n \times n$ matrix $[A|AB]$ into row reduced echelon form. Similar than the augmented matrix $[A|b]$ was brought into the form $[1|A^{-1}b]$, we end up with $[1|A^{-1}AB] = [1|B]$. By looking at the $n \times n$ matrix to the right during the Gauss elmination, the determinant has changed by $\det(A)$. We end up with a matrix B which has determinant $\det(B)$. Therefore, $\det(AB) = \det(A)\det(B)$.

PROBLEM. $A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 1 & 2 & 4 & 5 \\ 0 & 7 & 2 & 9 \\ 0 & 0 & 6 & 4 \end{bmatrix}$. Three transpositions of rows give $B = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 7 & 2 & 9 \\ 0 & 0 & 6 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ a matrix which has determinant 84. Therefore $\det(A) = (-1)^3 \det(B) = -84$.

PROBLEM. Determine $\det(A^{100})$, where A is the matrix $\begin{vmatrix} 1 & 2 \\ 3 & 16 \end{vmatrix}$.

SOLUTION. $\det(A) = 10$, $\det(A^{100}) = (\det(A))^{100} = 10^{100} = 1 \cdot$ gogool. This name as well as the gogoolplex $= 10^{10^{100}}$ are official. They are huge numbers: the mass of the universe for example is $10^{52} kg$ and $1/10^{10^{51}}$ is the chance to find yourself on Mars by quantum fluctuations. (R.E. Crandall, Scient. Amer., Feb. 1997).

ROW REDUCED ECHELON FORM. Determining rref(A) also determines $\det(A)$.
If A is a matrix and α_i are the factors which are used to scale different rows and s is the number of times, two rows are switched, then $\det(A) = (-1)^s \alpha_1 \cdots \alpha_n \det(\text{rref}(A))$.

INVERTIBILITY. A $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.

THE LAPLACE EXPANSION.
In order to calculate by hand the determinant of $n \times n$ matrices $A = a_{ij}$ for $n > 3$, the following expansion is useful. Choose a column i. For each entry a_{ji} in that column, take the $(n-1) \times (n-1)$ matrix A_{ij} called **minor** which does not contain the i'th column and j'th row. Then

$$\det(A) = (-1)^{i+1} a_{i1} \det(A_{i1}) + \cdots + (-1)^{i+n} a_{in} \det(A_{in}) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}) \ .$$

VAN DER MONDE DETERMINANT.
If we define for a scalar a the vector $\vec{a} = \left[\ 1, a, a^2, \dots, a^n\ \right]$. For $n+1$ scalars $a_0, \dots, a_n$ we consider the $(n+1) \times (n+1)$-matrix with rows $\vec{a_i}$.
CLAIM: $\det(A) = \prod_{i>j}(a_i - a_j)$. To prove this, we make a Laplace expansion with respect to the last column. To do so, we call $a_n = x$ and see that the determinant is a polynomial $f(x)$ of degree n in x. It satisfies $f(a_0) = f(a_1) = ... = f(a_{n-1}) = 0$ because in those cases, the determinant is zero. Therefore $f(x) = k(x - a_1) \dots, (x - a_{n-1})$ for some constant k. The number k is the coefficient in front of x^n: it is the van der Monde determinant in the case $n-1$. The recursion $\mathrm{VM}_n = \prod_{n>j}(a_n - a_j)\mathrm{VM}_{n-1}$ proves the claim.

BRAIN TEASERS.
1) What is the determinant of

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ \pi & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi & 0 & 0 & 0 \\ 0 & 0 & 1 & \pi & 0 & 0 \\ 0 & 0 & 0 & 0 & \pi & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

2) Assume $A = -A^T$. What is $\det(A)$?

3) Assume a A has integer entries and A^{-1} has also integer entries, then $\det(A) = 1$ or $\det(A) = -1$.

4) What values can the determinant of an orthogonal matrix have? (Remember that it satisfies $AA^T = 1$.)

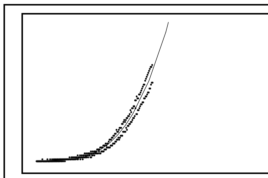
5) What is the determinant of $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 5 & 0 & 6 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & 5 & 1 \end{bmatrix}$. (Hint: Use the Laplace expansion).

6) (True or False) If a matrix has integer entries, the determinant is an integer.
(True or False) If the determinant is an integer, the matrix has integer entries.

(True or False) A matrix with positive entries has positive determinant.

QR DECOMPOSITION. Also the QR decomposition allows to compute the determinant. The determinant of Q is ± 1, the determinant of R is the product of the diagonal elements of that tridiagonal matrix.

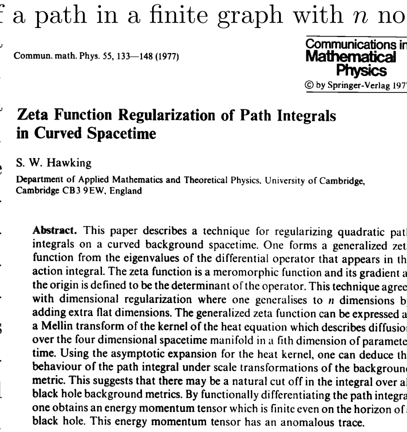
HOW FAST CAN ONE COMPUTE THE DETERMINANT?. Doing the Gauss elimination needs about n^3 steps. The cost to compute the determinant is therefore also of the order n^3.



The graph to the left shows the time Mathematica needs to calculate the determinant in dependence on the size of the $n \times n$ matrix. The matrix size goes from n=1 to n=300. The best cubic fit of these data has been obtained by the least square method from chapter IV.

DETERMINANTS IN PHYSICS. Physicists are excited about determinants because summing over all possible "paths" is used as a quantization method. The Feynmann path integral is a "summation" over a suitable set of paths and leads to quantum mechanics. What does it have to do with determinants? Each summand of the determinant can be interpreted like a contribution of a path in a finite graph with n nodes.

The article of Hawking deals with a determinant functional in physics. This Ray-Singer determinant is a number attached to a geometry rsp. an infinite dimensional matrix (the Laplacian) attached to the geometry. Physicists trying to glue quantum mechanics with general relativity hope to make sense of expressions like $\int_g e^{i\det(A(g))}$, where A is an operator attached to some geometry g and where the integral "sums" over all possible geometries.



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Zeta Function Regularization of Path Integrals in Curved Spacetime

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Abstract. This paper describes a technique for regularizing quadratic path integrals on a curved background spacetime. One forms a generalized zeta function from the eigenvalues of the differential operator that appears in the action integral. The zeta function is a meromorphic function and its gradient at the origin is defined to be the determinant of the operator. This technique agrees with dimensional regularization where one generalises to n dimensions by adding extra flat dimensions. The generalized zeta function can be expressed as a Mellin transform of the kernel of the heat equation which describes diffusion over the four dimensional spacetime manifold in a fith dimension of parameter time. Using the asymptotic expansion for the heat kernel, one can deduce the behaviour of the path integral under scale transformations of the background metric. This suggests that there may be a natural cut off in the integral over all black hole background metrics. By functionally differentiating the path integral one obtains an energy momentum tensor which is finite even on the horizon of a black hole. This energy momentum tensor has an anomalous trace.