Orthogonal Matrices 3/12/2002 — Math 21b, O. Knill

HOMEWORK: 5.3: 2,6,8,18*,20,44defgh*

DEFINITION The **transpose** of a matrix A is the matrix $(A^T)_{ij} = A_{ji}$. If A is a $n \times m$ matrix, then A^T is a $m \times n$ matrix. For square matrices, the transposed matrix is obtained by reflecting the matrix at the diagonal.

EXAMPLES The transpose of a vector $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is the row vector $A^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$.

The transpose of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

A PROPERTY OF THE TRANSPOSE.
a) If x, y are two vectors, then $x \cdot Ay = A^T x \cdot y$.
b) $(AB)^T = B^T A^T$.
c) $(A^T)^T = A$.

PROOFS. a) Because $x \cdot Ay = \sum_j \sum_i x_i A_{ij} y_j$ and $A^T x \cdot y = \sum_j \sum_i A_{ji} x_i y_j$ the two expressions are the same by renaming i and j.
b) $(AB)_{kl} = \sum_i A_{ki} B_{il}$. $(AB)^T_{kl} = \sum_i A_{li} B_{ik} = A^T B^T$.
c) $((A^T)^T)_{ij} = (A^T)_{ji} = A_{ij}$.

DEFINITION. A $n \times n$ matrix A is called **orthogonal** if $A^T A = 1$. The corresponding linear transformation is called **orthogonal**.

INVERSE. It is easy to invert an orthogonal matrix: $A^{-1} = A^T$.

EXAMPLES. The rotation matrix $A = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix}$ is orthogonal because its column vectors have length 1 and are orthogonal to each other. Indeed: $A^T A = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix} \cdot \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. A reflection at a line is an orthogonal transformation because the columns of the matrix A have length 1 and are orthogonal. Indeed: $A^T A = \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix} \cdot \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

FACTS. An orthogonal transformation preserves the dot product: $A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y}$ Proof: this is a homework assignment: Hint: just look at the properties of the transpose.

Orthogonal transformations preserve the **length** of vectors as well as the **angles** between them.

Proof. We have $||A\vec{x}||^2 = A\vec{x} \cdot A\vec{x} = \vec{x} \cdot \vec{x}\ ||\vec{x}||^2$. Let α be the angle between $\vec{x}$ and $\vec{y}$ and let β denote the angle between $A\vec{x}$ and $A\vec{y}$ and α the angle between $\vec{x}$ and $\vec{y}$. Using $A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y}$ we get $||A\vec{x}||||A\vec{y}|| \cos(\beta) = A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y} = ||\vec{x}||||\vec{y}|| \cos(\alpha)$. Because $||A\vec{x}|| = ||\vec{x}||, ||A\vec{y}|| = ||\vec{y}||$, this means $\cos(\alpha) = \cos(\beta)$. Because this property holds for all vectors we can rotate $\vec{x}$ in plane V spanned by $\vec{x}$ and $\vec{y}$ by an angle ϕ to get $\cos(\alpha + \phi) = \cos(\beta + \phi)$ for all ϕ. Differentiation with respect to ϕ at $\phi = 0$ shows also $\sin(\alpha) = \sin(\beta)$ so that $\alpha = \beta$.

ORTHOGONAL MATRICES AND BASIS. A linear transformation A is orthogonal if and only if the column vectors of A form an orthonormal basis. (That is what $A^T A = 1_n$ means.)

COMPOSITION OF ORTHOGONAL TRANSFORMATIONS. The composition of two orthogonal transformations is orthogonal. The inverse of an orthogonal transformation is orthogonal. Proof. The properties of the transpose give $(AB)^T AB = B^T A^T AB = B^T B = 1$ and $(A^{-1})^T A^{-1} = (A^T)^{-1} A^{-1} = (AA^T)^{-1} = 1$.

EXAMPLES.
The composition of two reflections at a line is a rotation.
The composition of two rotations is a rotation.
The composition of a reflections at a plane with a reflection at an other plane is a rotation (the axis of rotation is the intersection of the planes).

ORTHOGONAL PROJECTIONS. The orthogonal projection P onto a linear space with orthonormal basis $\vec{v}_1, \dots, \vec{v}_n$ is the matrix AA^T, where A is the matrix with column vectors $\vec{v}_i$.
To see this just translate the formula $P\vec{x} = (\vec{v}_1 \cdot \vec{x})\vec{v}_1 + \dots + (\vec{v}_n \cdot \vec{x})\vec{v}_n$ into the language of matrices: $A^T\vec{x}$ is a vector with components $\vec{b}_i = (\vec{v}_i \cdot \vec{x})$ and $A\vec{b}$ is the sum of the $\vec{b}_i\vec{v}_i$, where $\vec{v}_i$ are the column vectors of A.

EXAMPLE. Find the orthogonal projection P from $\mathbf{R}^3$ to the linear space spanned by $\vec{v}_1 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \frac{1}{5}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Solution: $AA^T = \begin{bmatrix} 0 & 1 \\ 3/5 & 0 \\ 4/5 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3/5 & 4/5 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 9/25 & 12/25 \\ 0 & 12/25 & 16/25 \end{bmatrix}$.

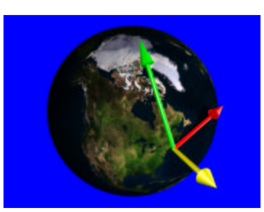
WHY DO WE CARE ABOUT ORTHOGONAL TRANSFORMATIONS?

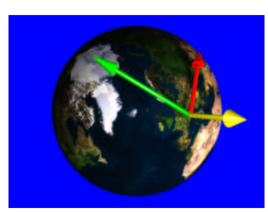
- In Physics, Galileo transformations are compositions of translations with orthogonal transformations. The laws of classical mechanics are invariant under such transformations. This is a symmetry.
- Many coordinate transformations are orthogonal transformations. We will see examples when dealing with differential equations.
- In the QR decomposition of a matrix A, the matrix Q is orthogonal. Because $Q^{-1} = Q^t$, this allows to invert A easier.
- Fourier transformations are orthogonal transformations. We will see this transformation later in the course. In application, it is useful in computer graphics (i.e. JPG), sound compression (i.e. MP3).
- Quantum mechanical evolutions (when written as real matrices) are orthogonal transformations.

WHICH OF THE FOLLOWING MAPS ARE ORTHOGONAL TRANSFORMATIONS?:

Yes	No	Shear in the plane.
Yes	No	Projection in three dimensions onto a plane.
Yes	No	Reflection in two dimensions at the origin.
Yes	No	Reflection in three dimensions at a plane.
Yes	No	Dilation with factor 2.
Yes	No	The Lorenz boost $\vec{x} \mapsto A\vec{x}$ in the plane with $A = \begin{bmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{bmatrix}$
Yes	No	A translation.

CHANGING COORDINATES ON THE EARTH. Problem: what is the matrix which rotates a point on earth with (latitude,longitude)=(a_1, b_1) to a point with (latitude,longitude)=(a_2, b_2)? Solution: The matrix which rotate the point $(0,0)$ to (a,b) a composition of two rotations. The first rotation brings the point into the right latitude, the second brings the point into the right longitude. $R_{a,b} = \begin{bmatrix} \cos(b) & -\sin(b) & 0 \\ \sin(b) & \cos(b) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(a) & 0 & -\sin(a) \\ 0 & 1 & 0 \\ \sin(a) & 0 & \cos(a) \end{bmatrix}$. To bring a point (a_1, b_1) to a point (a_2, b_2), we form $A = R_{a_2,b_2} R_{a_1,b_1}^{-1}$.





Example: With Cambridge (USA): $(a_1, b_1) = (42.366944, 288.893889)\pi/180$ and Zürich (Switzerland): $(a_2, b_2) = (47.377778, 8.551111)\pi/180$, we get the matrix $A = \begin{bmatrix} 0.178313 & -0.980176 & -0.0863732 \\ 0.983567 & 0.180074 & -0.0129873 \\ 0.028284 & -0.082638 & 0.996178 \end{bmatrix}$.